International Research School of Water Resources

International PhD Course in
HYDROGEOPHYSICS

Inversion of Resistivity & IP data

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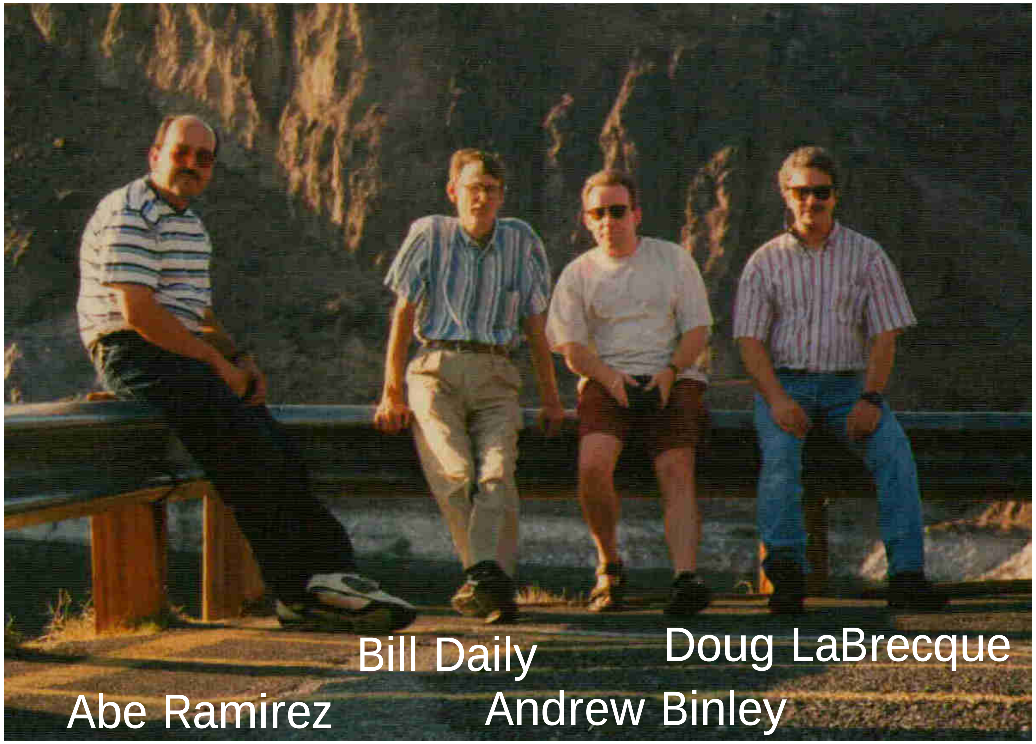
Notes supplied are based in part on

Binley & Kemna (2005) “DC Resistivity and Induced Polarization Methods”, In: *Hydrogeophysics* by Rubin, Y. and Hubbard, S (Eds.), Springer.

Daily, W., Ramirez, A., Binley, A. and LaBrecque, D. (2005) “Electrical Resistance Tomography - Theory and Practice”, in *Near Surface Geophysics* by D.K. Butler (Ed.), Investigations in Geophysics No. 13, Society of Exploration Geophysicists, p525-550.



Overview

We have shown how resistivity and IP can be measured in the field (and lab).

Here we present approaches for inversion of resistivity and IP data.

We cover:

- the basic principle of forward modelling resistivity and IP;
- inverse modelling of resistivity and IP;
- methods for image appraisal;
- practical guidelines

The examples used for cross-borehole but the same principles apply to surface and other configurations

Resistivity modelling

Forward Modelling -

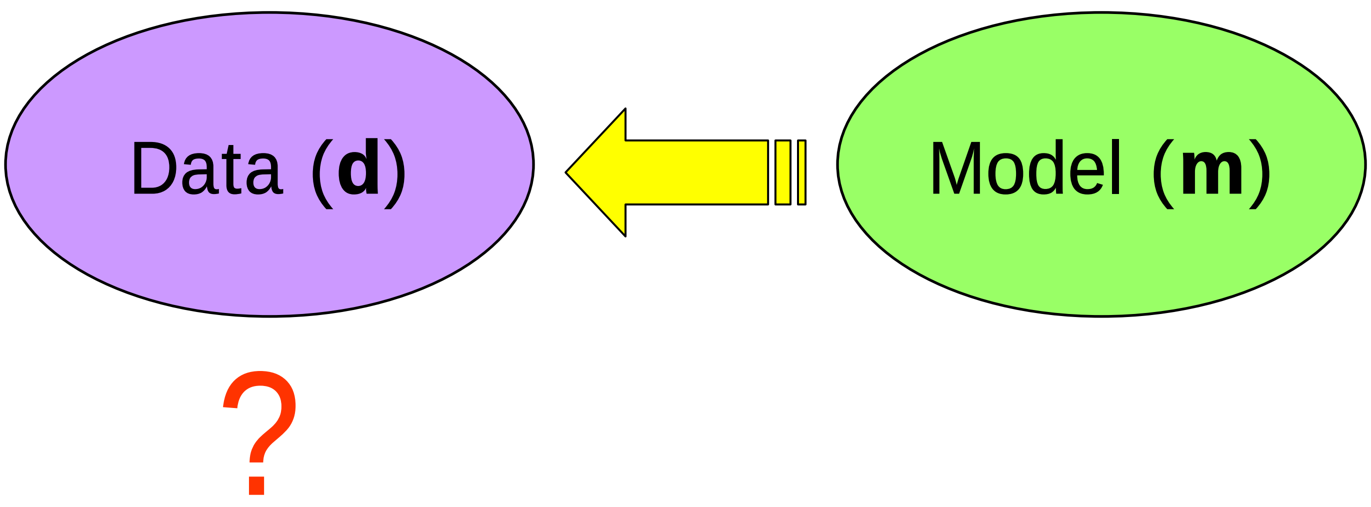
Calculating the resistances that would theoretically be ‘measured’ for a given resistivity distribution

Inverse Modelling -

Calculating the resistivity distribution that is ‘consistent’ with the observed (measured) resistances

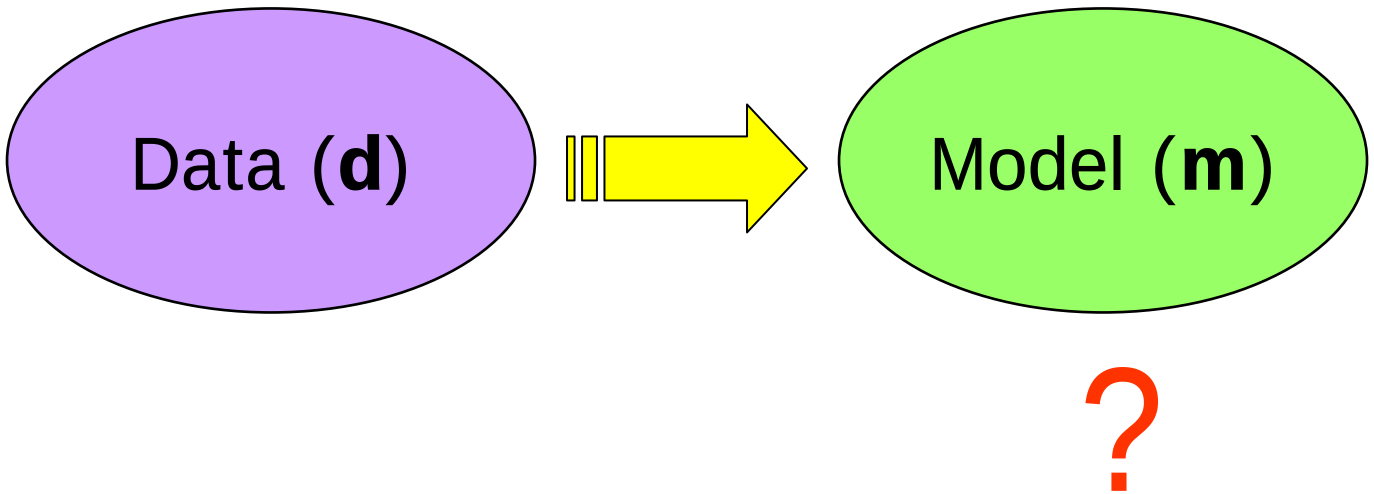
Forward Modelling -

Calculating the resistances that would theoretically be ‘measured’ for a given resistivity distribution



Inverse Modelling -

Calculating the resistivity distribution that is 'consistent' with the observed (measured) resistances



Resistivity Forward modelling

For a given distribution of conductivity $\sigma(x, y, z)$ we can determine the potentials from the solution of:

$$\frac{\partial}{\partial x}\left(\sigma \frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\sigma \frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\sigma \frac{\partial V}{\partial z}\right) = -I\,\delta(x)\delta(y)\delta(z)$$

with appropriate boundary conditions.

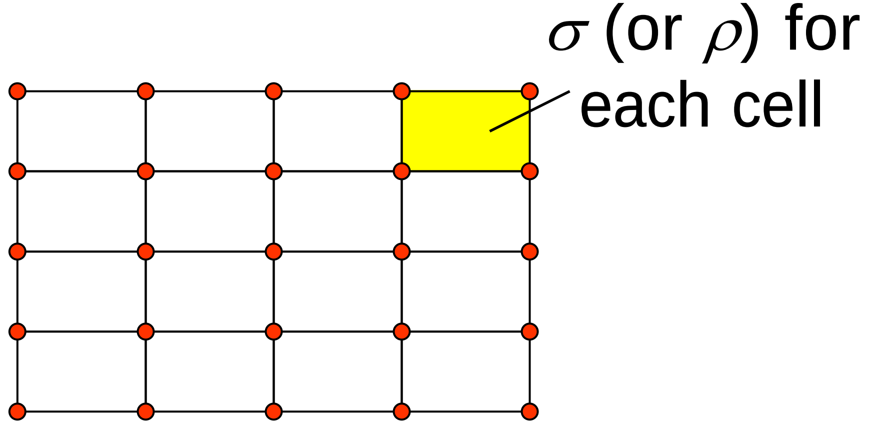
If the problem is considered to be 2-D, i.e. $\sigma = \sigma(x, z)$ then we require the solution of:

$$\frac{\partial}{\partial x}\left(\sigma \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\sigma \frac{\partial v}{\partial z}\right) - \lambda^2 \sigma v = -I\delta(x)\delta(z)$$

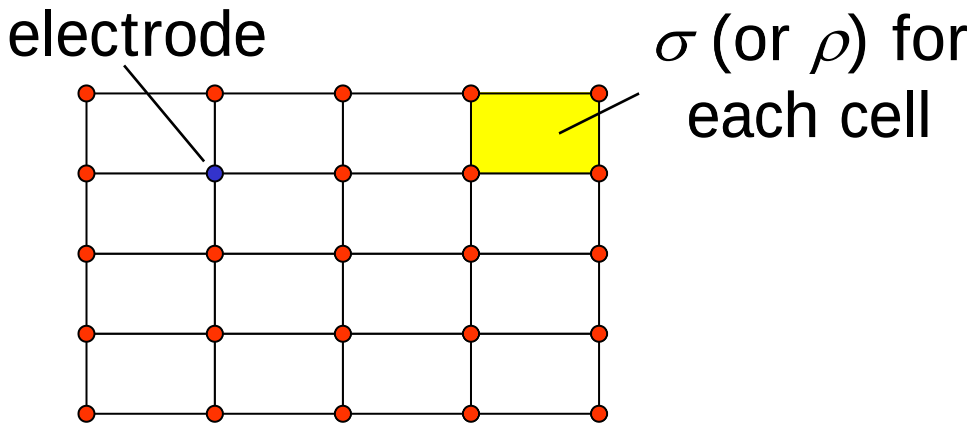
where λ is the Fourier-transform variable corresponding to the strike direction y, and v is the potential in the Fourier domain.

Finite difference and finite element methods are widely used for 2-D and 3-D solutions.

The region is discretised into cells (or elements) with nodes defining cell corners. A conductivity may be assigned to each cell and potentials determined at each node.



Electrode are located at node points and hence potentials may be determined at electrodes.



The forward equations are solved for current injection at each electrode.

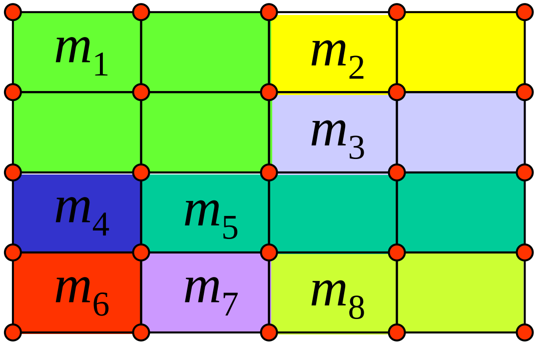
By superposition the four electrode resistances can be computed.

For large 3-D problems computational demands can be restrictive, for example with 150 electrodes and a grid with 50 x 50 x 100 nodes one complete forward solution requires the 150 solutions of for 250,000 unknowns.

Resistivity Inverse modelling

To solve the inverse problem the region is discretised into *parameters* **m** (usually log resistivities).

The parameters are single cells or groups of cells.



First we must define an objective function which we wish to minimise

We could use the data misfit:

$$\Psi_d = \sum_{i=1}^{N} \left(\frac{F_i(\mathbf{m}) - d_i}{\varepsilon_i} \right)^2 = \left\| W_d \left(F(\mathbf{m}) - d_i \right) \right\|^2$$

$F_i(\boldsymbol{m})$ is resistance for measurement i,
d_i is the i^{th} observed resistance,
ε_i is error for measurement i,
W_d is a matrix of errors ε,
N is number of measurements

By searching for the minimum of the objective function we can determine the 'best' set of parameters ***m***

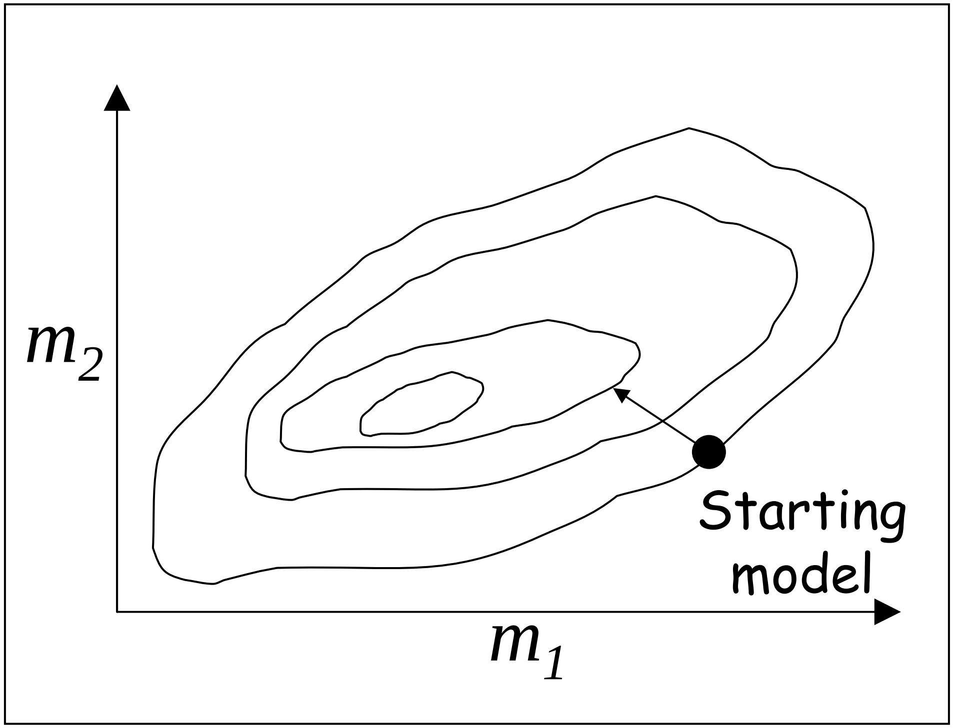
For the DC resistivity problem this must be done in an iterative manner …

1. Take initial estimate of resistivity in all cells

2. Compute step change in resistivity for all cells

3. Update resistivity in all cells

4. If not at acceptable level of misfit go to step 2



The problem with just using the data misfit as the objective function is that we will often have an

undetermined system -

too many unknowns (cell resistivities)
and
too few equations (measurements)

In addition, the solution can be very sensitive to data errors and may tend to give unrealistic results

What we need is some other way of *constraining* the inversion so that what we get makes sense (in a geophysical, hydrological or geological manner)

The most common approach is to introduce a *penalty* to the objective function so that the inversion does not give images that are too smooth or too different from some specified model, that is, *a priori* information is used

The objective function can then incorporate terms like:

$\Psi_m =$ Penalty for deviation from specified resistivity m_0 + Penalty for roughness in x and y direction

$$\psi_m = \alpha_s \left\| W_s (m - m_0) \right\|^2$$

$$+ \alpha_x \left\| W_x (m - m_0) \right\|^2 + \alpha_y \left\| W_y (m - m_0) \right\|^2$$

The total objective function is then:

$$\Psi = \Psi_d + \Psi_m$$

$$= \left\| W_d \left(F(\mathbf{m}) - d \right) \right\|^2 + \alpha \left\| W_m (m - m_0) \right\|^2$$

Where W_m incorporates relative contribution of each of the penalty terms

This objective function can be minimised in order to determine the 'best' values of $\mathbf{m}$

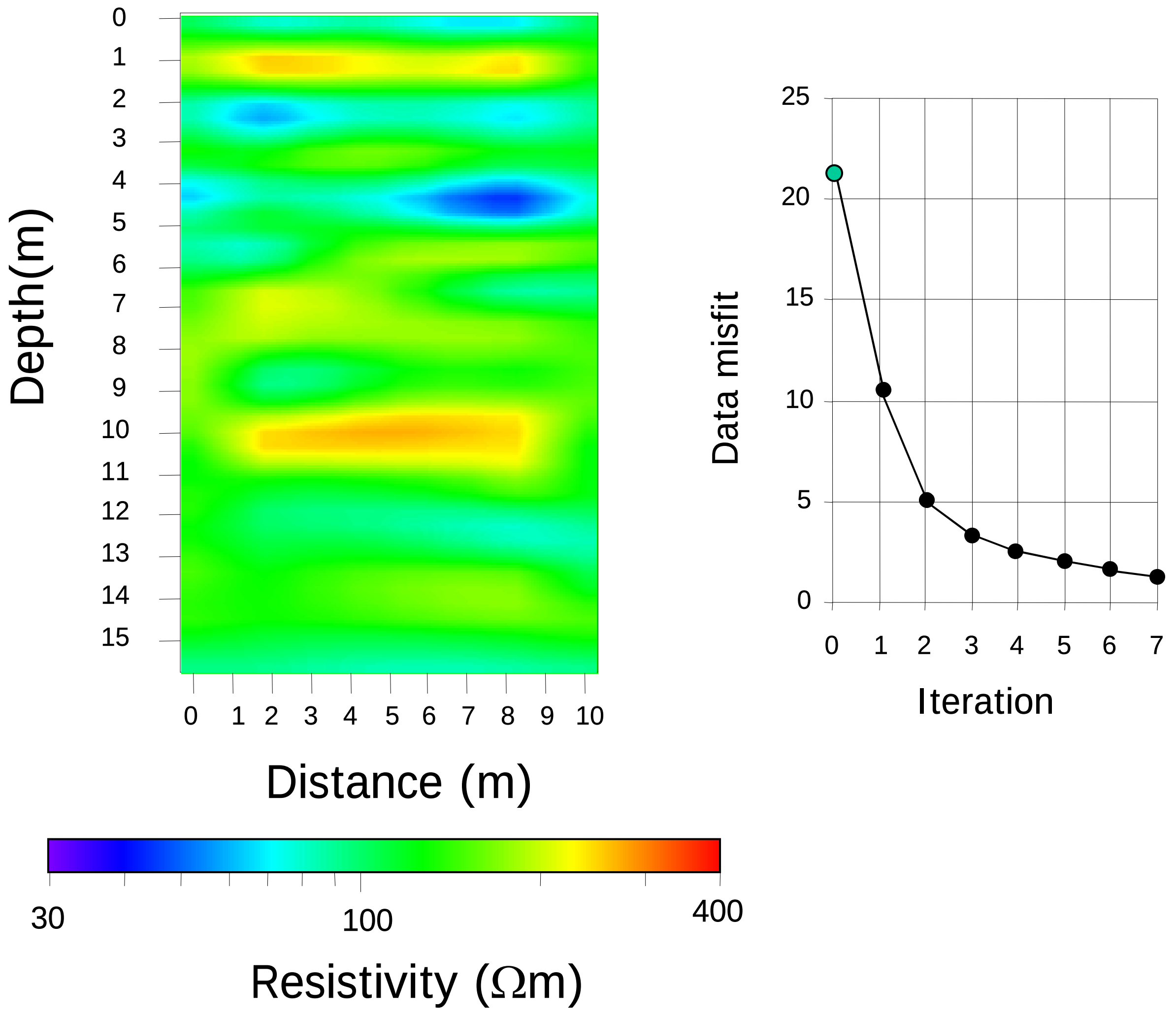
$$\left(\mathbf{J}_k{}^T \mathbf{W}_\mathrm{d}{}^T \mathbf{W}_\mathrm{d} \mathbf{J}_k + \alpha \mathbf{W}_\mathrm{m}{}^\mathrm{T} \mathbf{W}_\mathrm{m}\right) \Delta\mathbf{m}_k = \mathbf{J}_k{}^T \mathbf{W}_\mathrm{d}{}^T \mathbf{W}_\mathrm{d} \left[\mathbf{d} - F(\mathbf{m}_k)\right] - \alpha \mathbf{W}_\mathrm{m}{}^\mathrm{T} \mathbf{W}_\mathrm{m} \left(\mathbf{m}_k - \mathbf{m}_0\right)$$

$$\mathbf{m}_{k+1} = \mathbf{m}_k + \Delta\mathbf{m}_k \qquad k = 1,2,3,\ldots$$

$\mathbf{J}$ is the Jacobian (or sensitivity) matrix

Normally α is optimised at each iteration step but some inversion programs fix α

Example



Depth(m)
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
0 1 2 3 4 5 6 7 8 9 10
Distance (m)
30
100
400
Resistivity (Ωm)
Data misfit
25
20
15
10
5
0
0 1 2 3 4 5 6 7
Iteration

Dealing with noisy data -

The data you collect and the forward models you use in the inverse solution will have errors.

It is essential that you determine these errors.

It is not appropriate to run the inversion until either

(a) you exceed a specified number of iterations or

(b) no improvement is seen between two successive iterations.

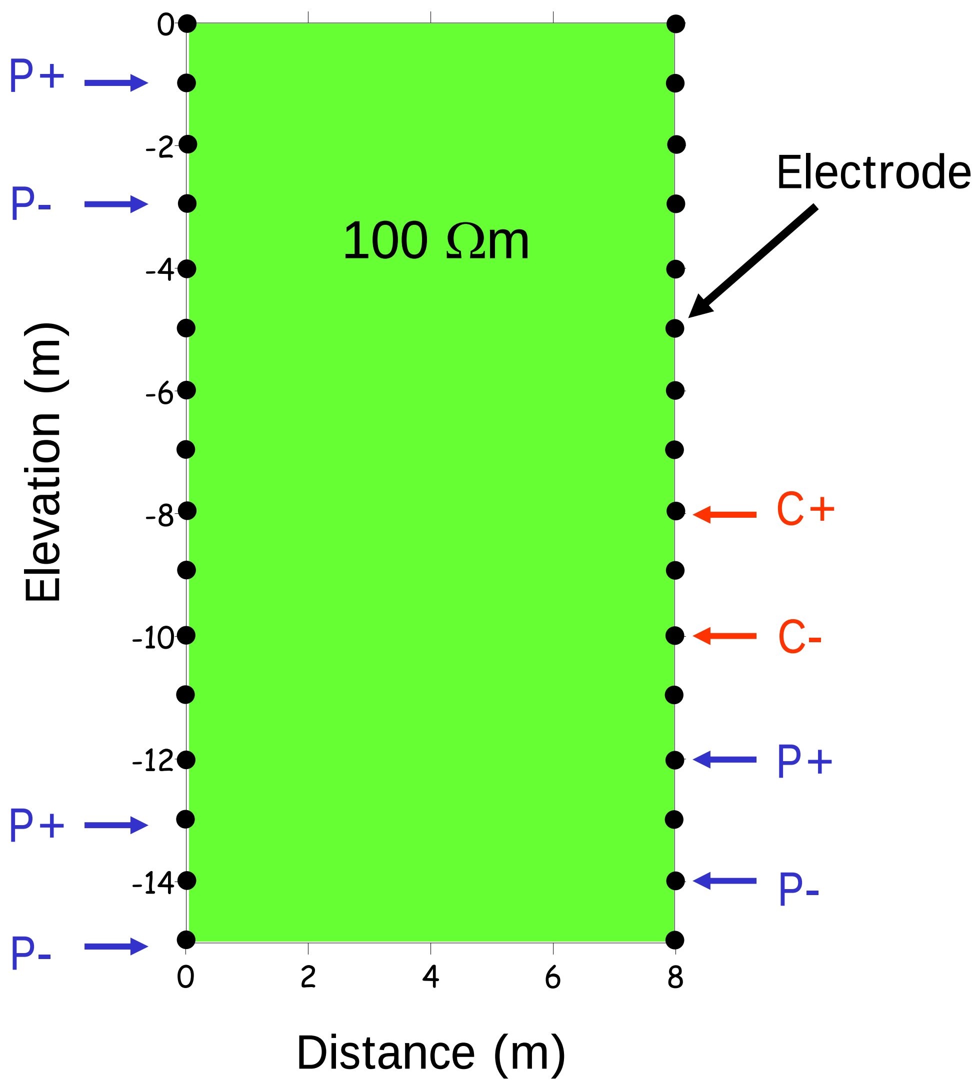
We can illustrate these concepts with some synthetic model experiments …

Forward Modelling Errors

Definition of problem

‘Skip 1’ schedule used, i.e. dipole-dipole with 2 electrode spacing dipole length

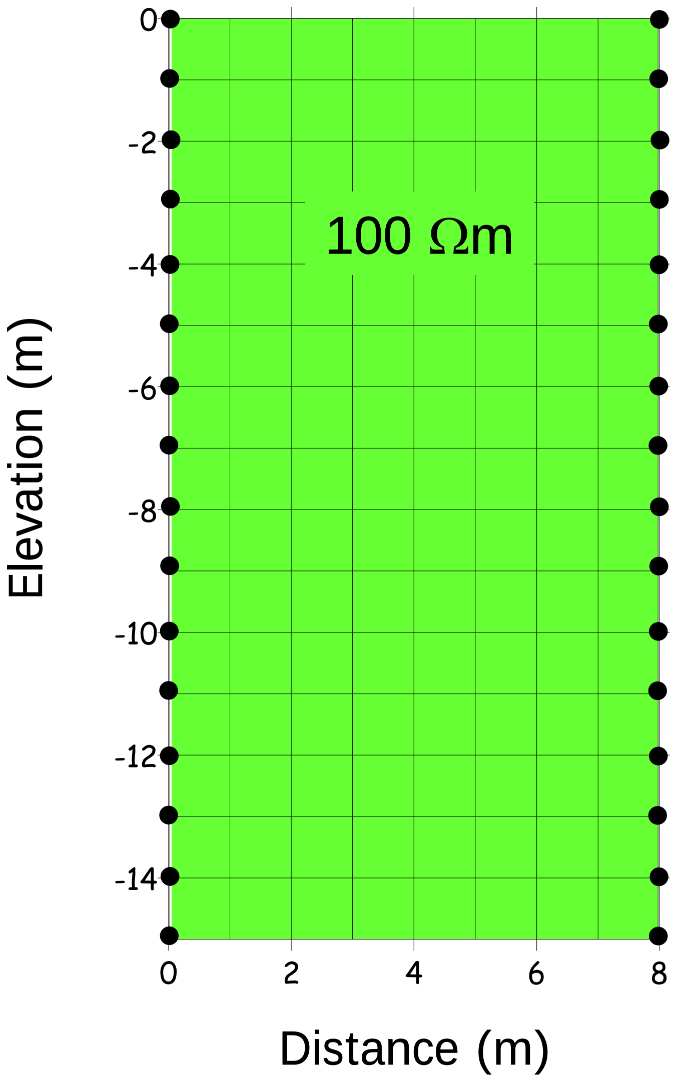
Total of 405 measurements



Forward Modelling Errors

Definition of problem

Mesh 1:
Each finite element is 1 m x 1m, i.e. the electrode spacing



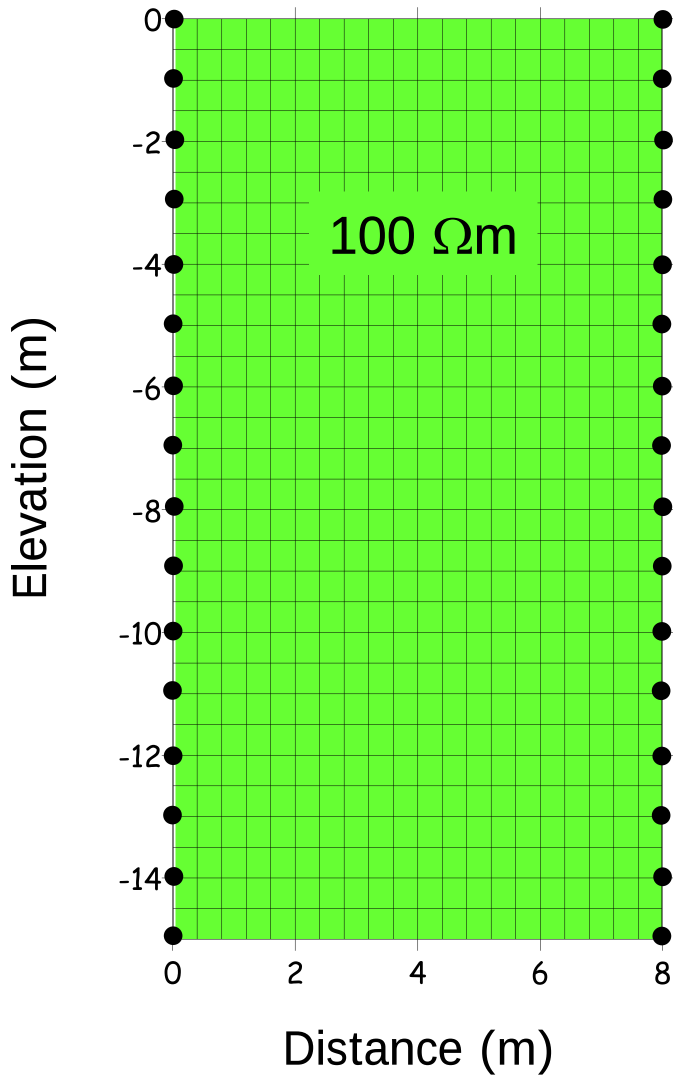
Note:
The mesh extends out left, right and down to account for infinite current paths

Errors:
> 3% error: 108
> 2% error: 205
> 1% error: 359

Forward Modelling Errors

Definition of problem

Mesh 2:
Each finite element is 0.5 m x 0.5m, i.e. half an electrode spacing



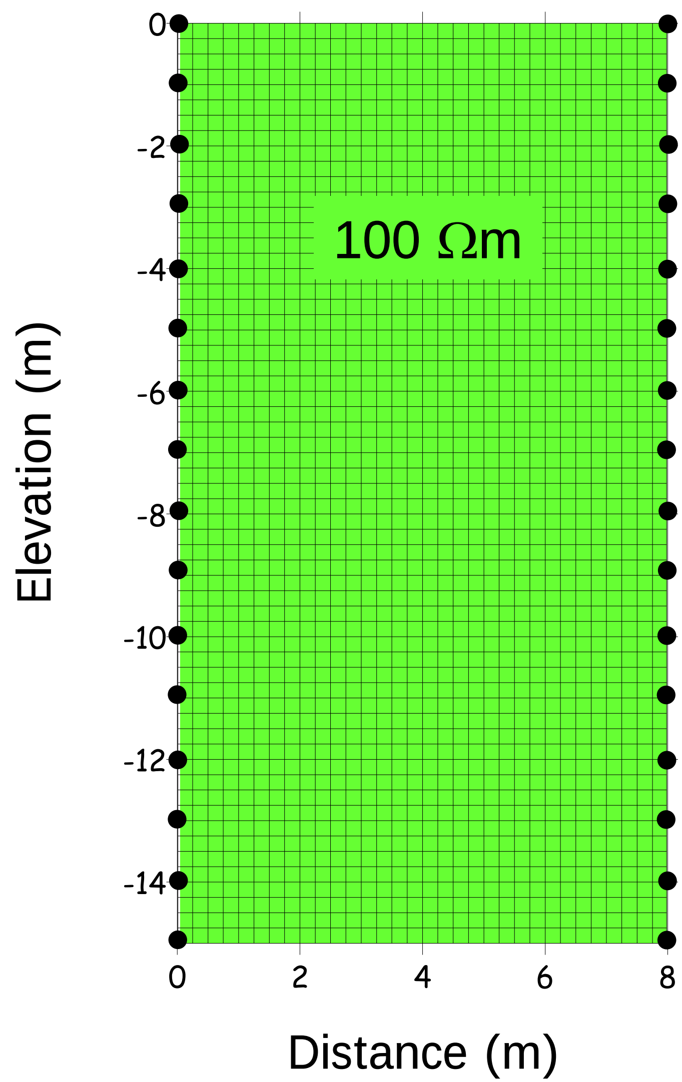
Note:
The mesh extends out left, right and down to account for infinite current paths

Errors:
> 3% error: 11
> 2% error: 34
> 1% error: 138

Forward Modelling Errors

Definition of problem

Mesh 3:
Each finite element is 0.25 m x 0.25m, i.e. quarter an electrode spacing

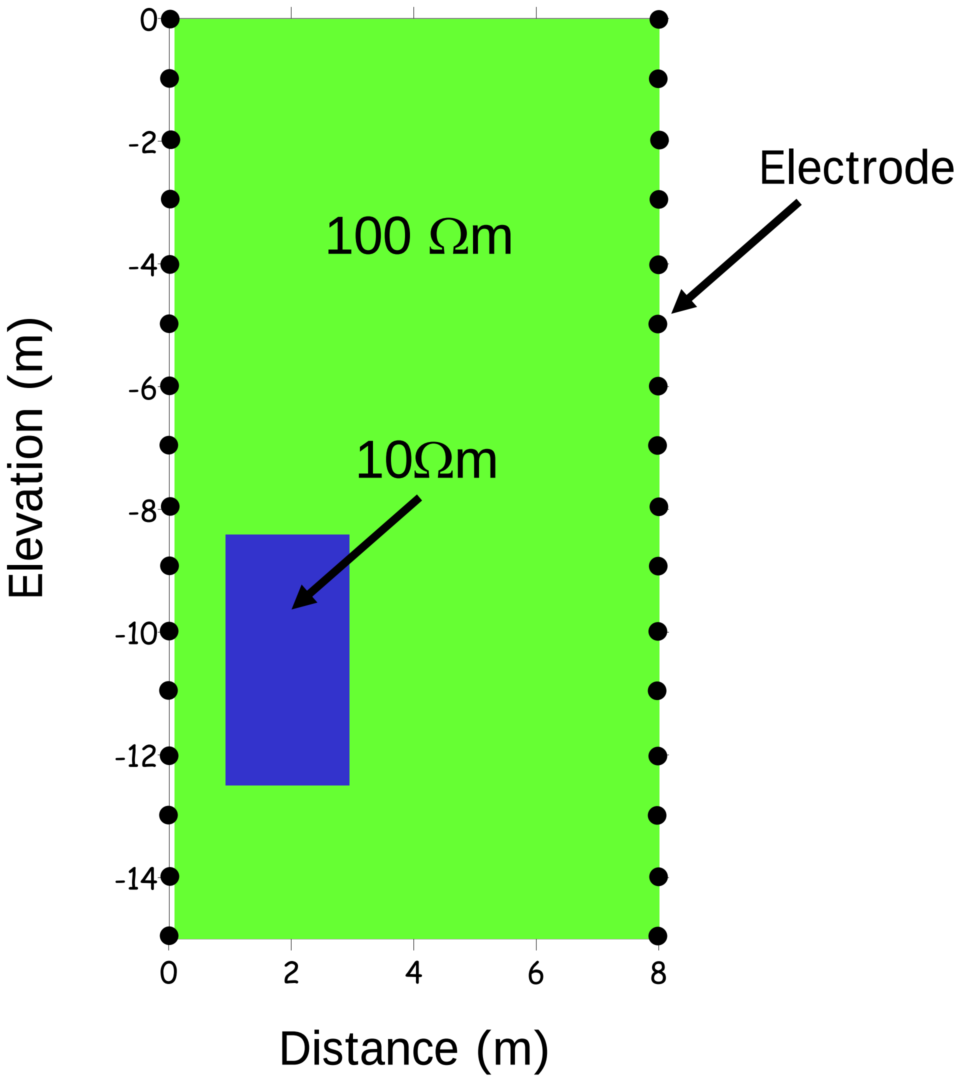


Note:
The mesh extends out left, right and down to account for infinite current paths

Errors:
> 3% error: 4
> 2% error: 8
> 1% error: 54

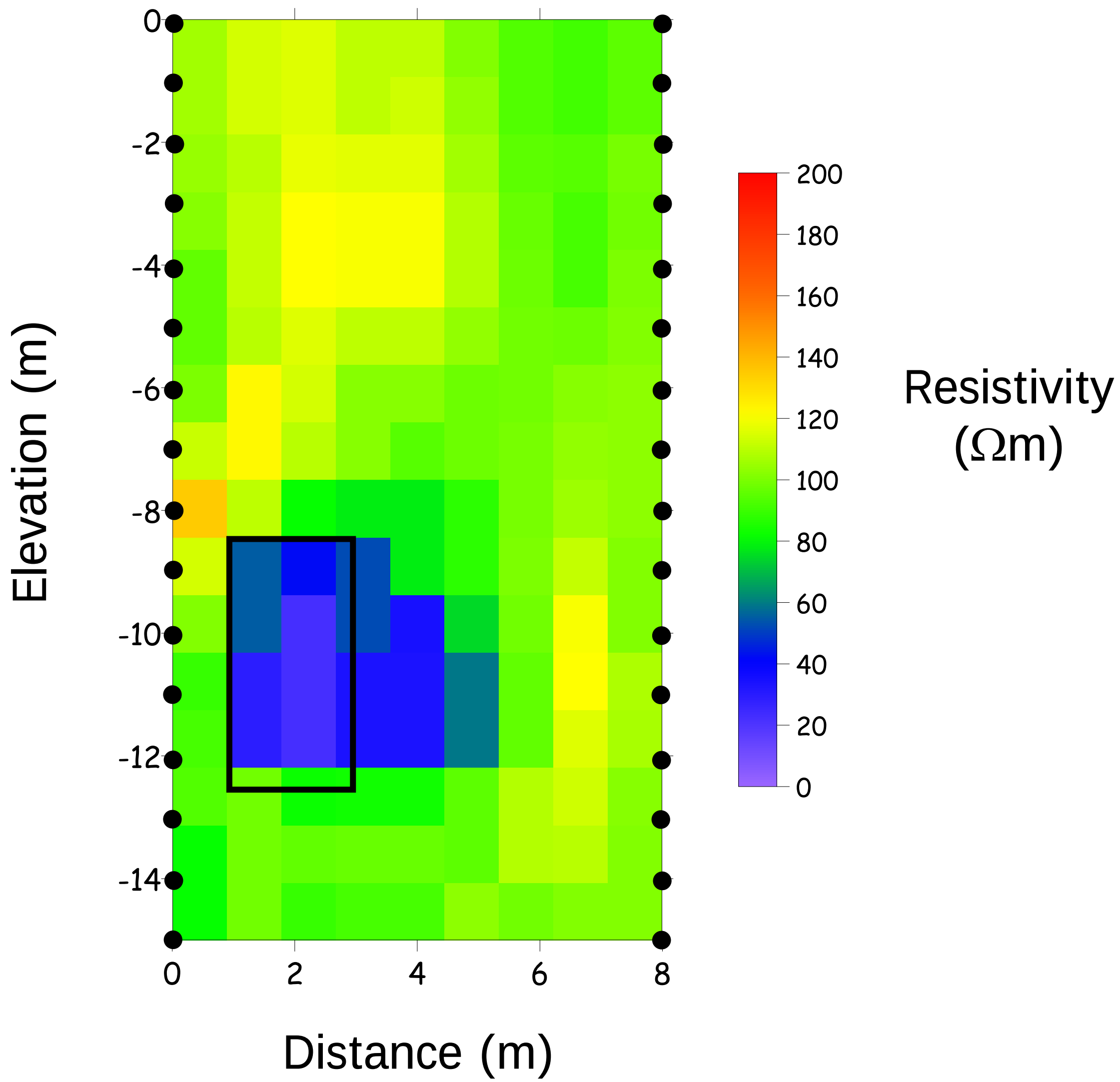
Synthetic Data Test

Using Mesh 3



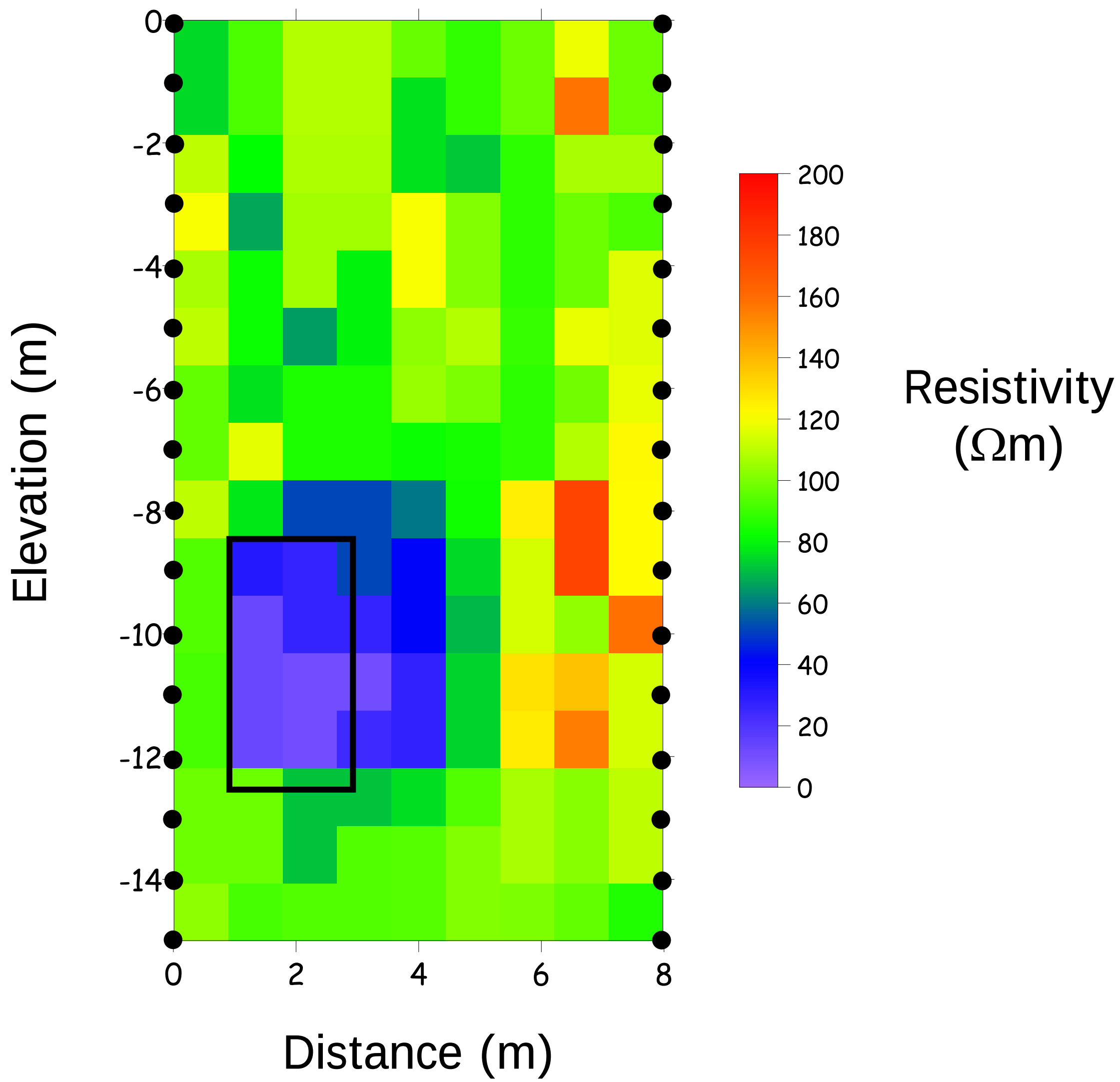
Inversion of ‘noise free’ data

(but good to 2% based on forward model error)



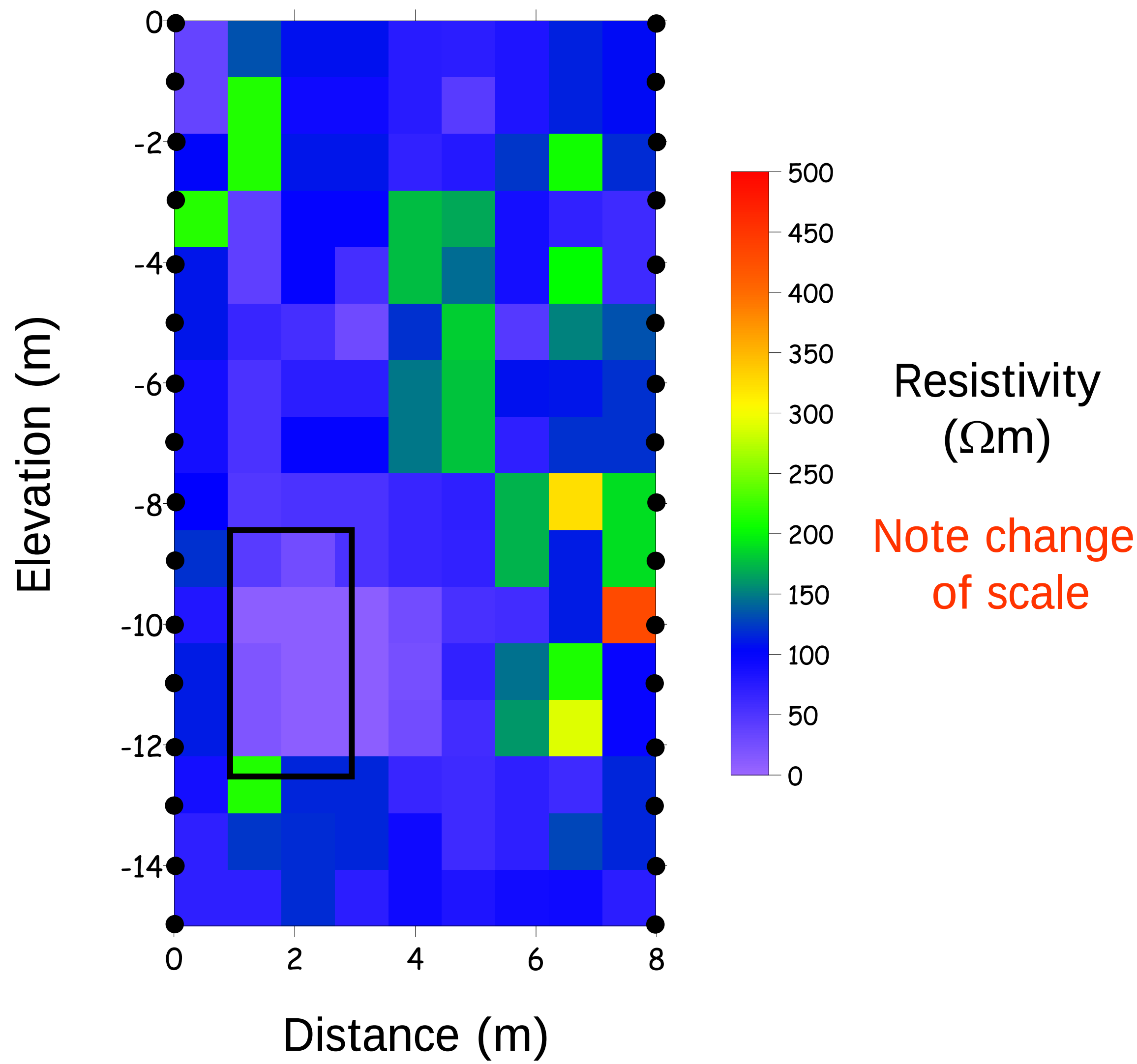
Inversion of data with 5% noise added

(assuming for the inversion we have 2% error)



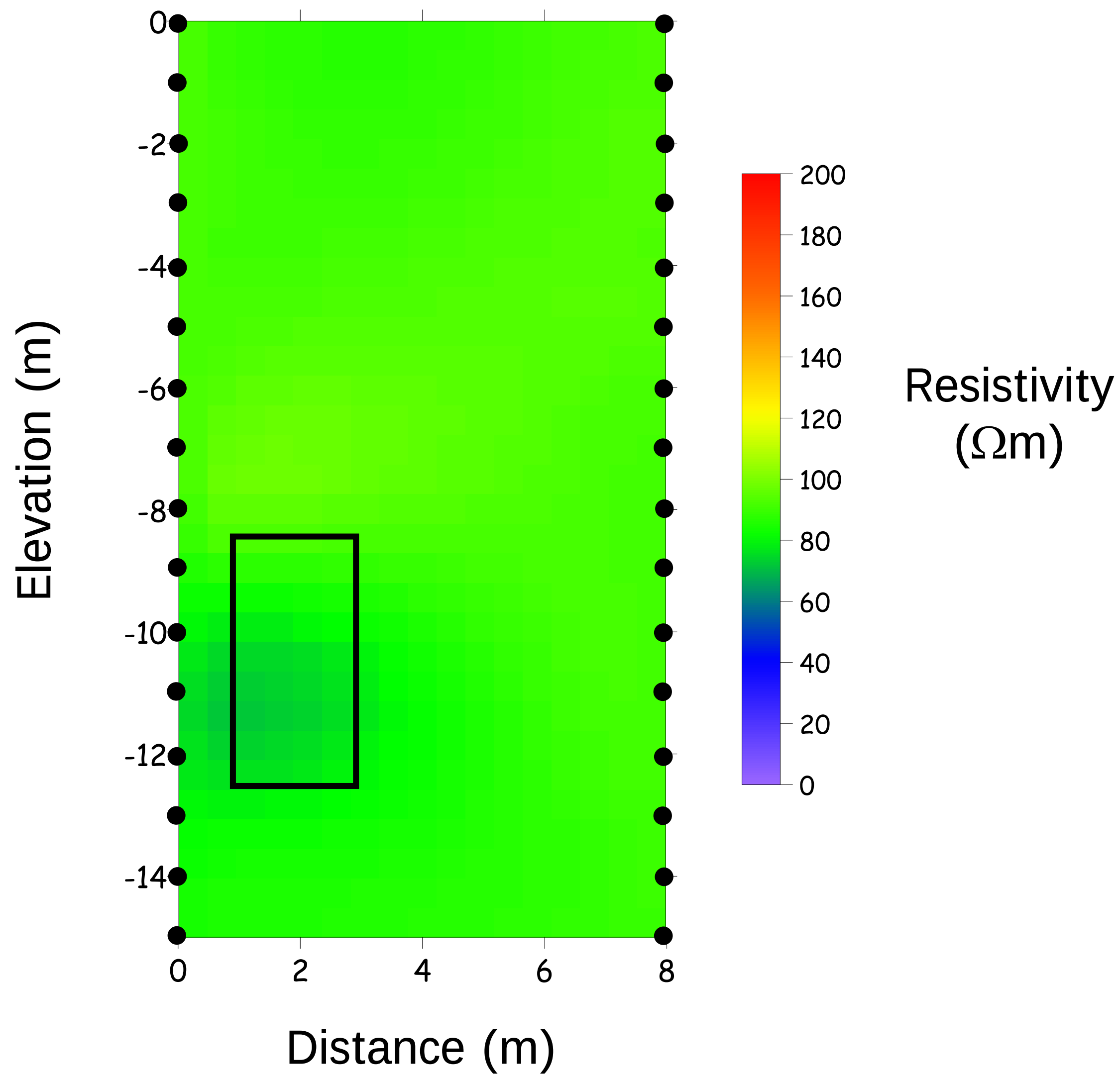
Inversion of data with 10% noise added

(assuming for the inversion we have 2% error)



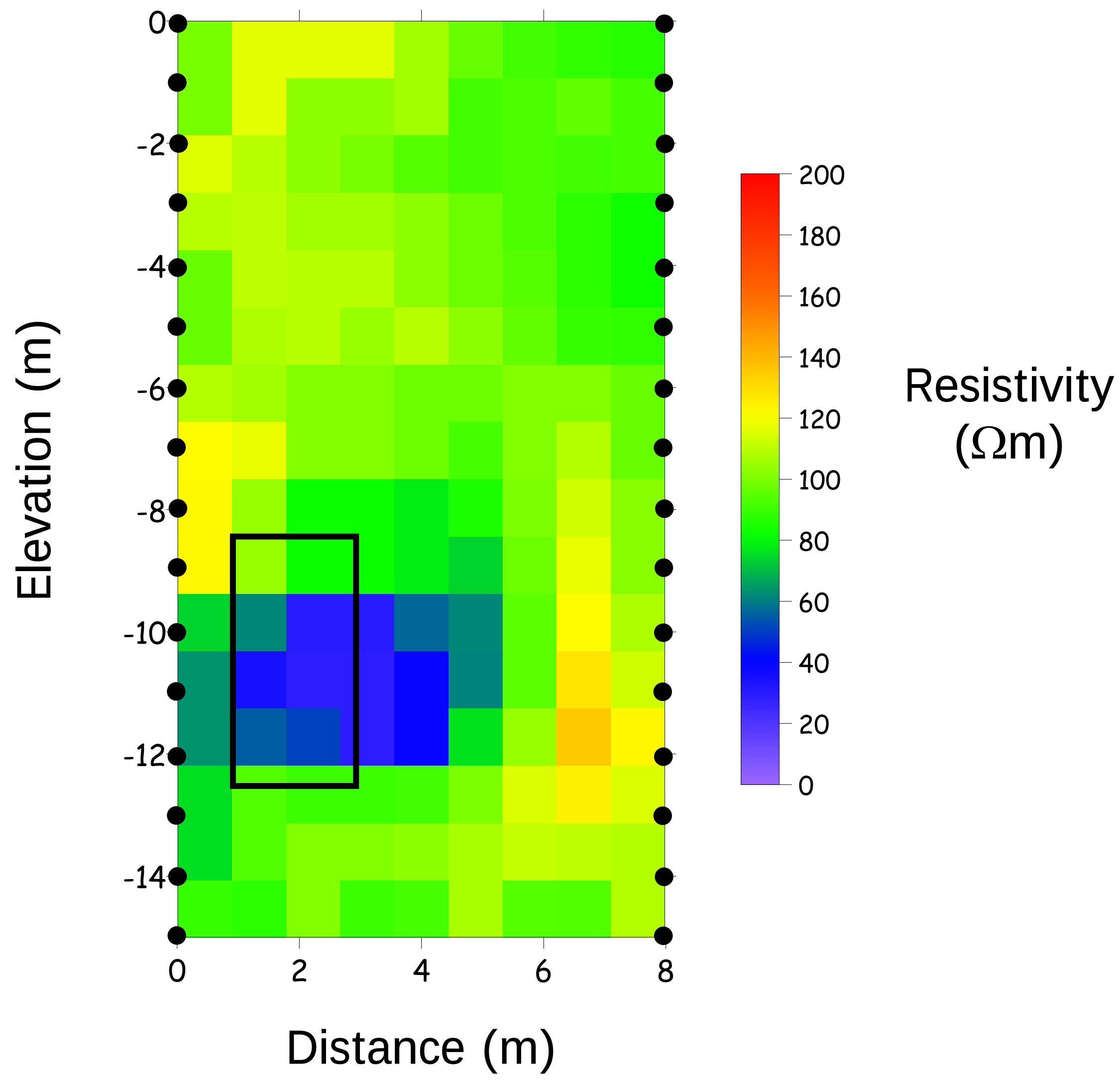
Inversion of data with 10% noise added

(assuming for the inversion we have 20% error)



Inversion of data with 10% noise added

(assuming for the inversion we have 10% error)



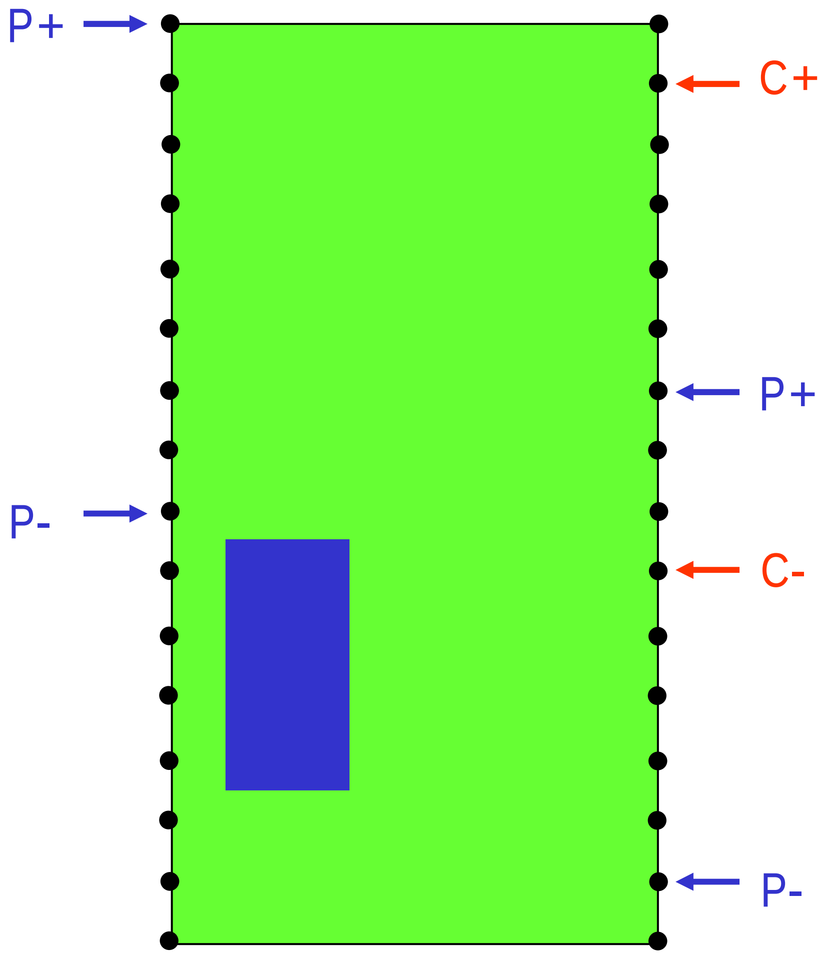
What these results show -

- Make sure you know how good the forward model is.
- Use at least 2 cells (elements) between electrodes (more if you can).
- The above will allow you to assess errors in your model but you must also assess the errors of your data

How do different measurement schemes compare ?

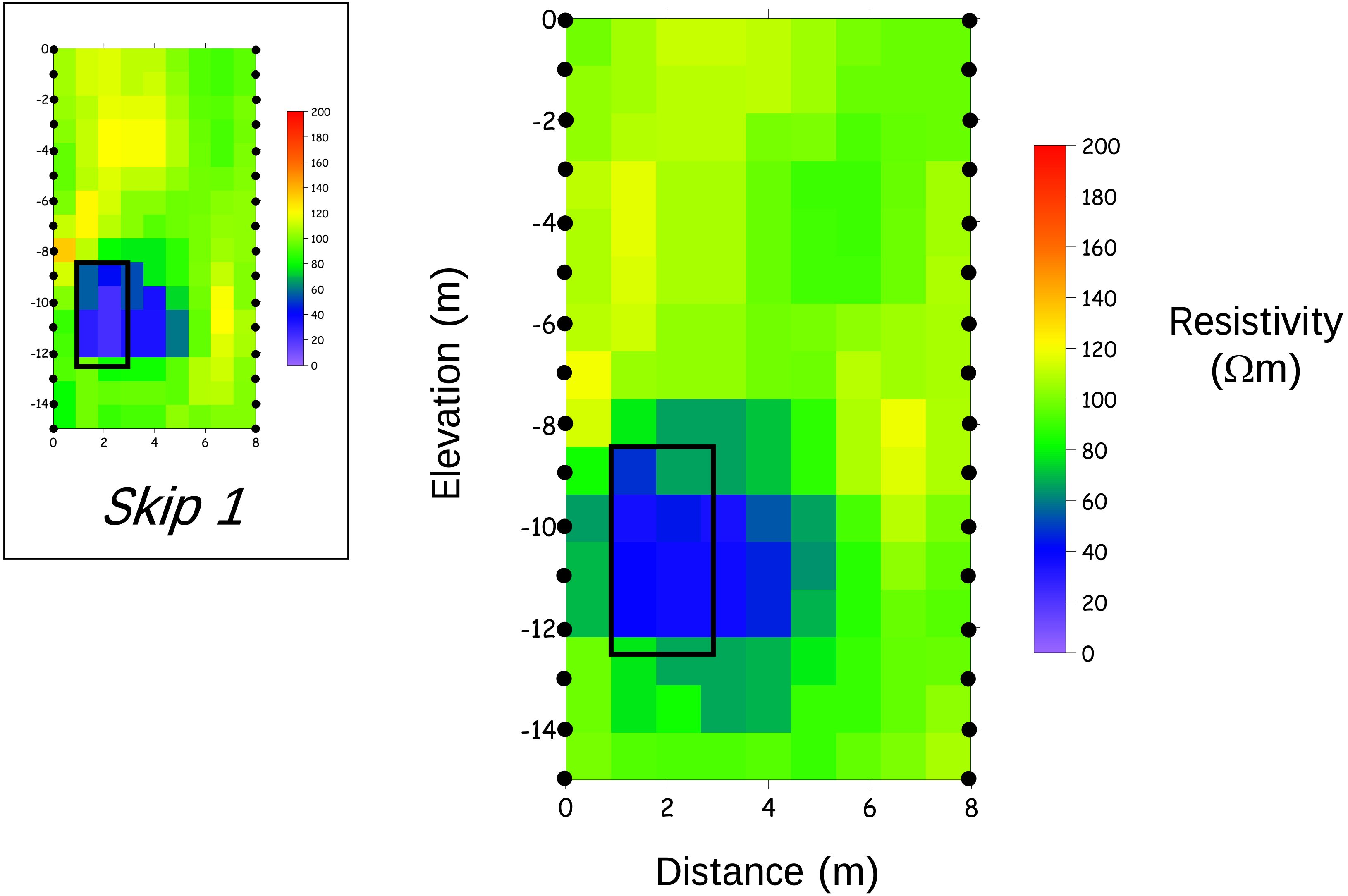
Inversion of ‘noise free’ data using *Skip 7*

(assuming for the inversion we have 2% error)



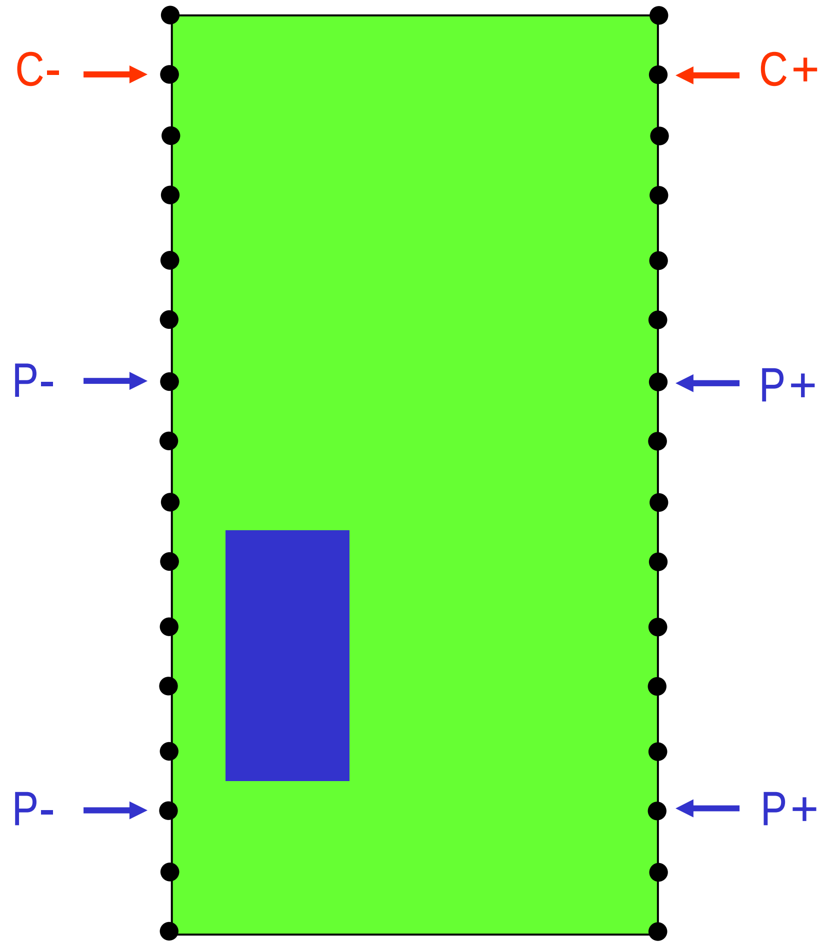
Inversion of ‘noise free’ data using *Skip 7*

(assuming for the inversion we have 2% error)



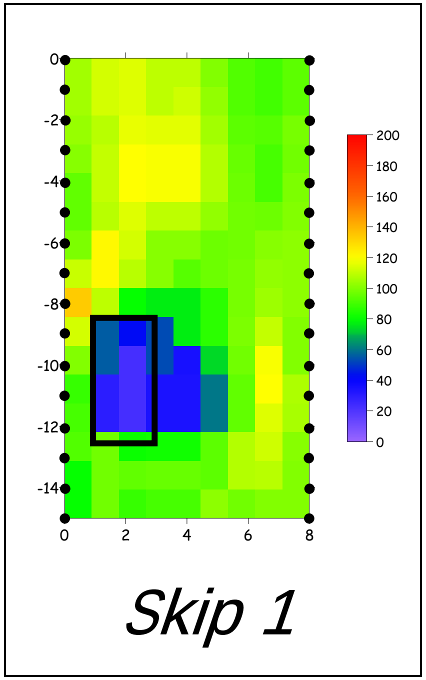
Inversion of ‘noise free’ data using *Skip 15*

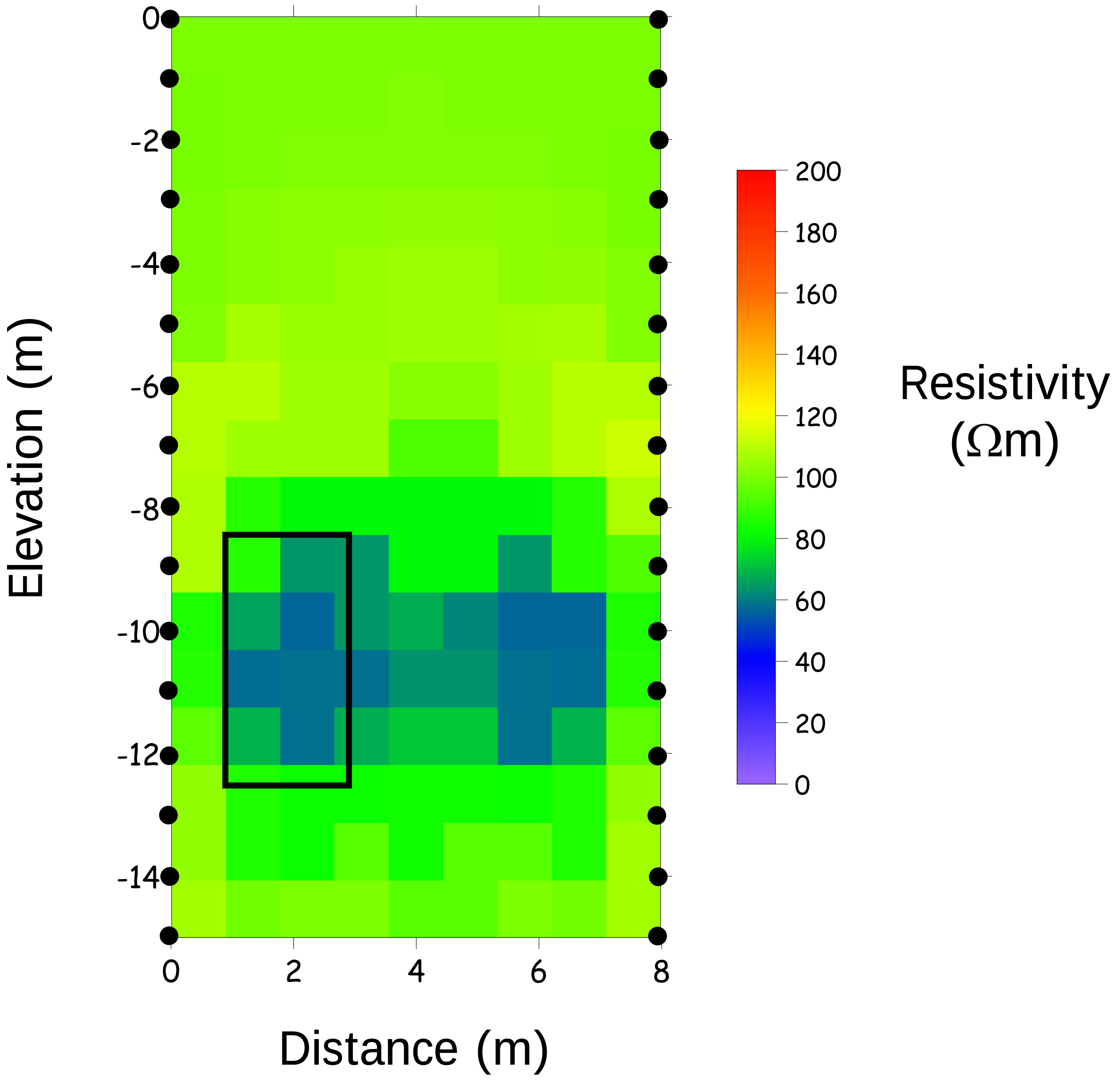
(assuming for the inversion we have 2% error)



Inversion of ‘noise free’ data using *Skip 15*

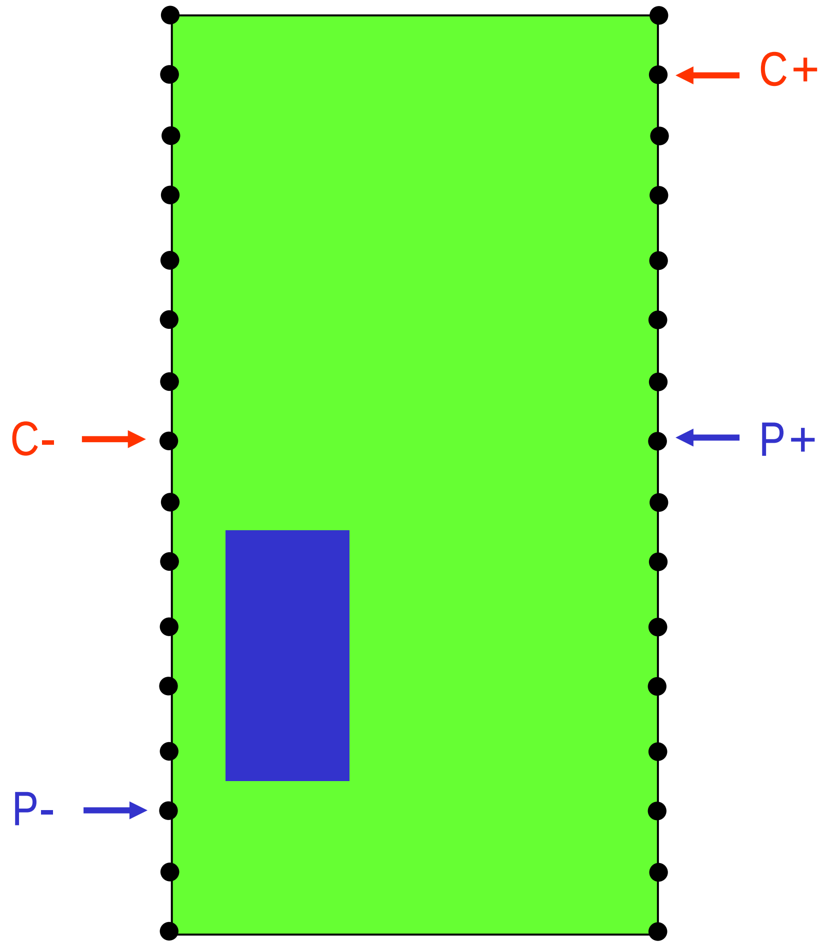
(assuming for the inversion we have 2% error)





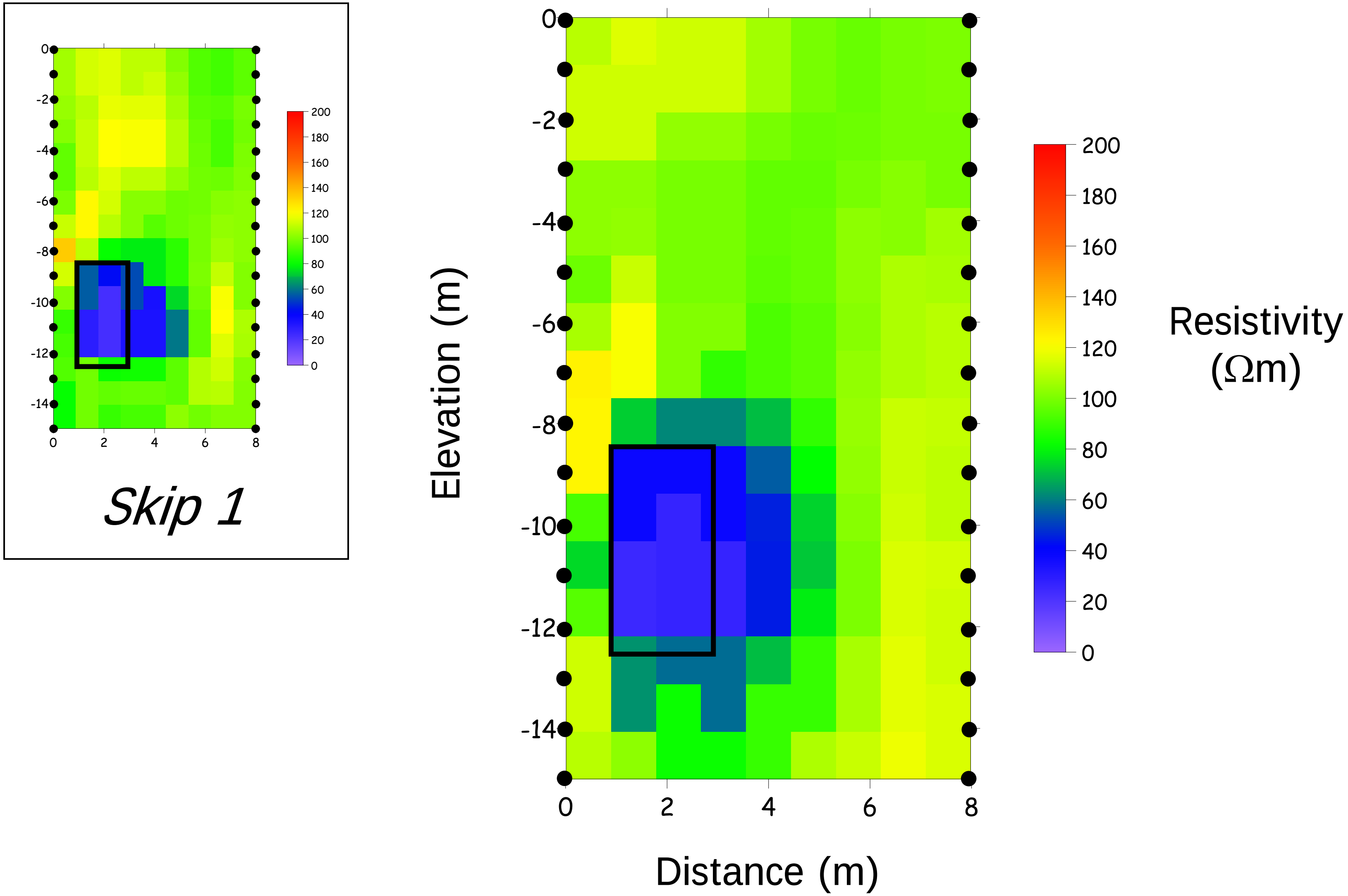
Inversion of 'noise free' data using *Skip 21*

(assuming for the inversion we have 2% error)



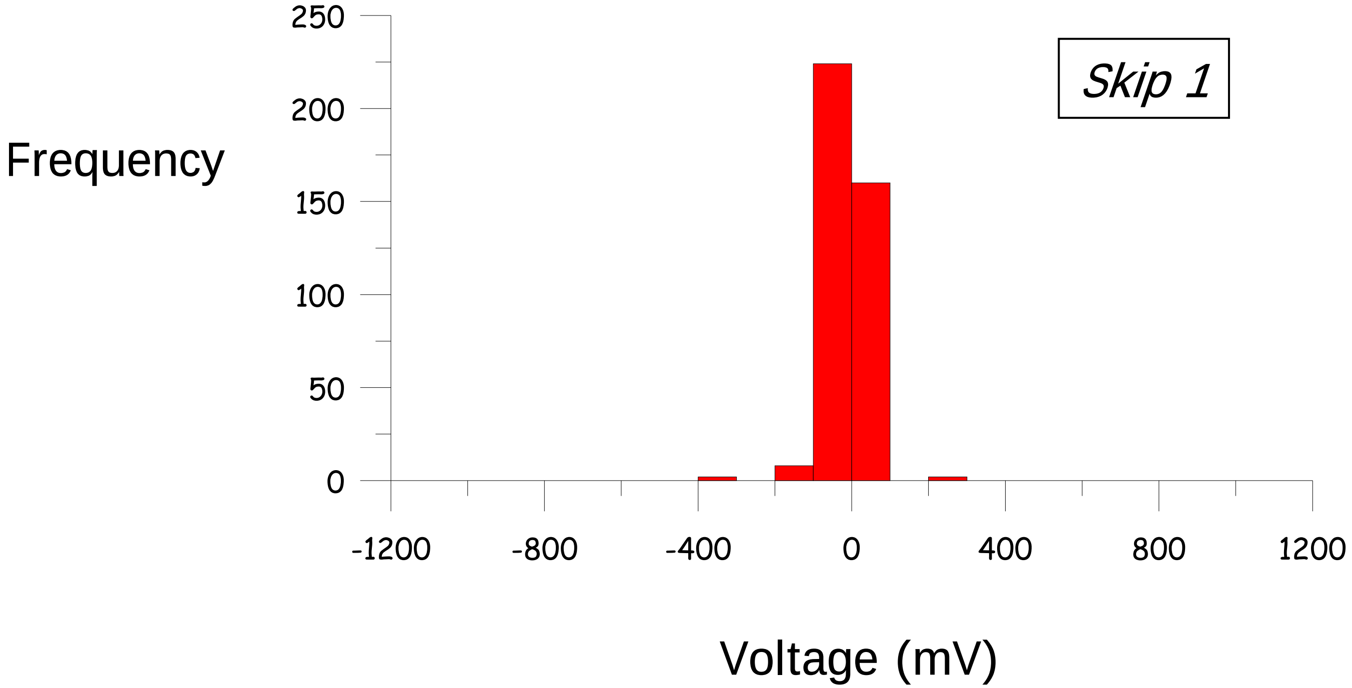
Inversion of 'noise free' data using *Skip 21*

(assuming for the inversion we have 2% error)

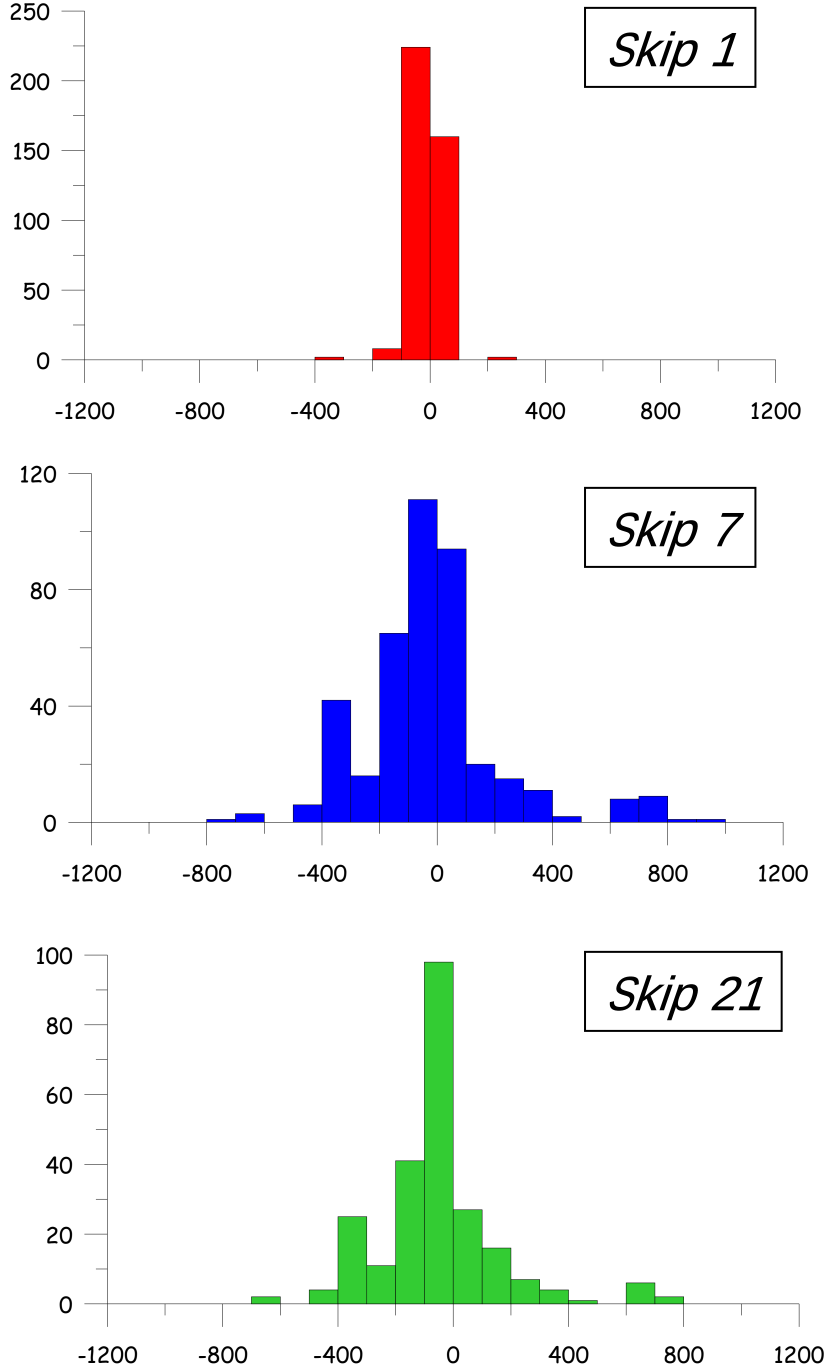


Distribution of voltages

(Assuming a constant current of 50 mA - you may do worse !)



Comparison for different measurement schemes

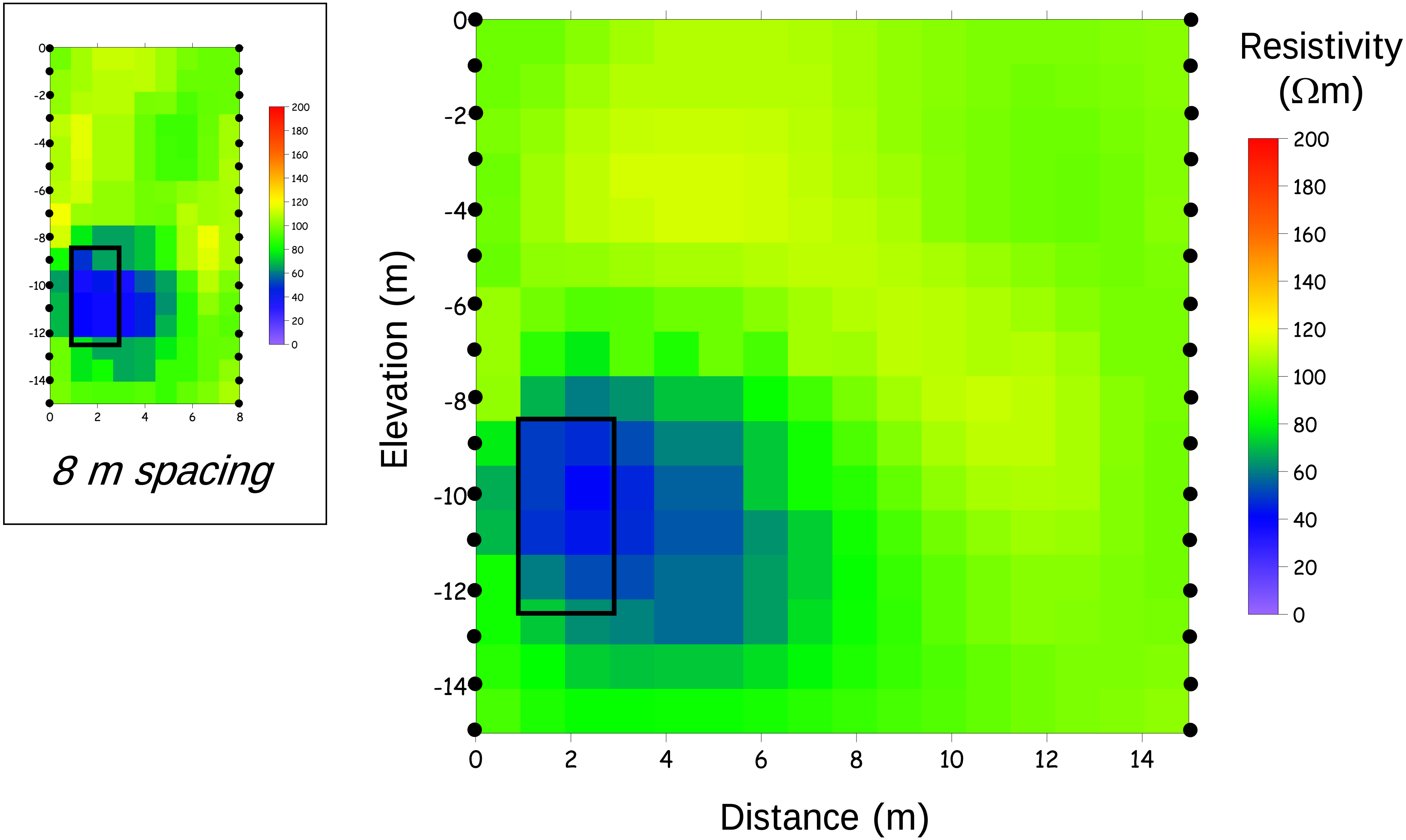


Some schemes may not be suitable for instruments with a poor dynamic range, particularly if a constant current source is used.

How does the borehole separation affect the ability the image resistivity ?

Inversion of ‘noise free’ data using *Skip 7*

(Borehole spacing now 15 m)



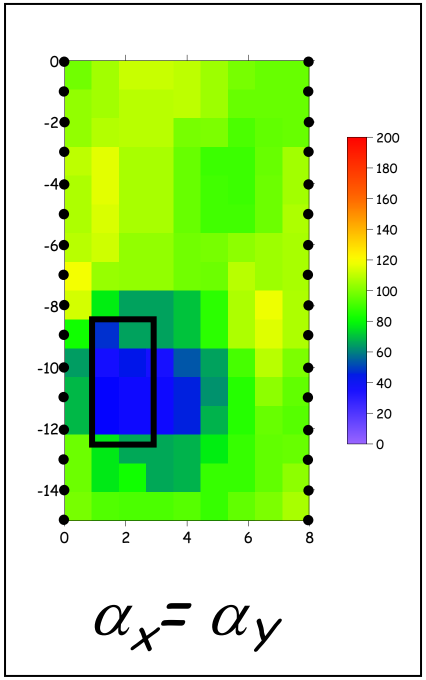
How does the borehole separation affect the ability the image resistivity ?

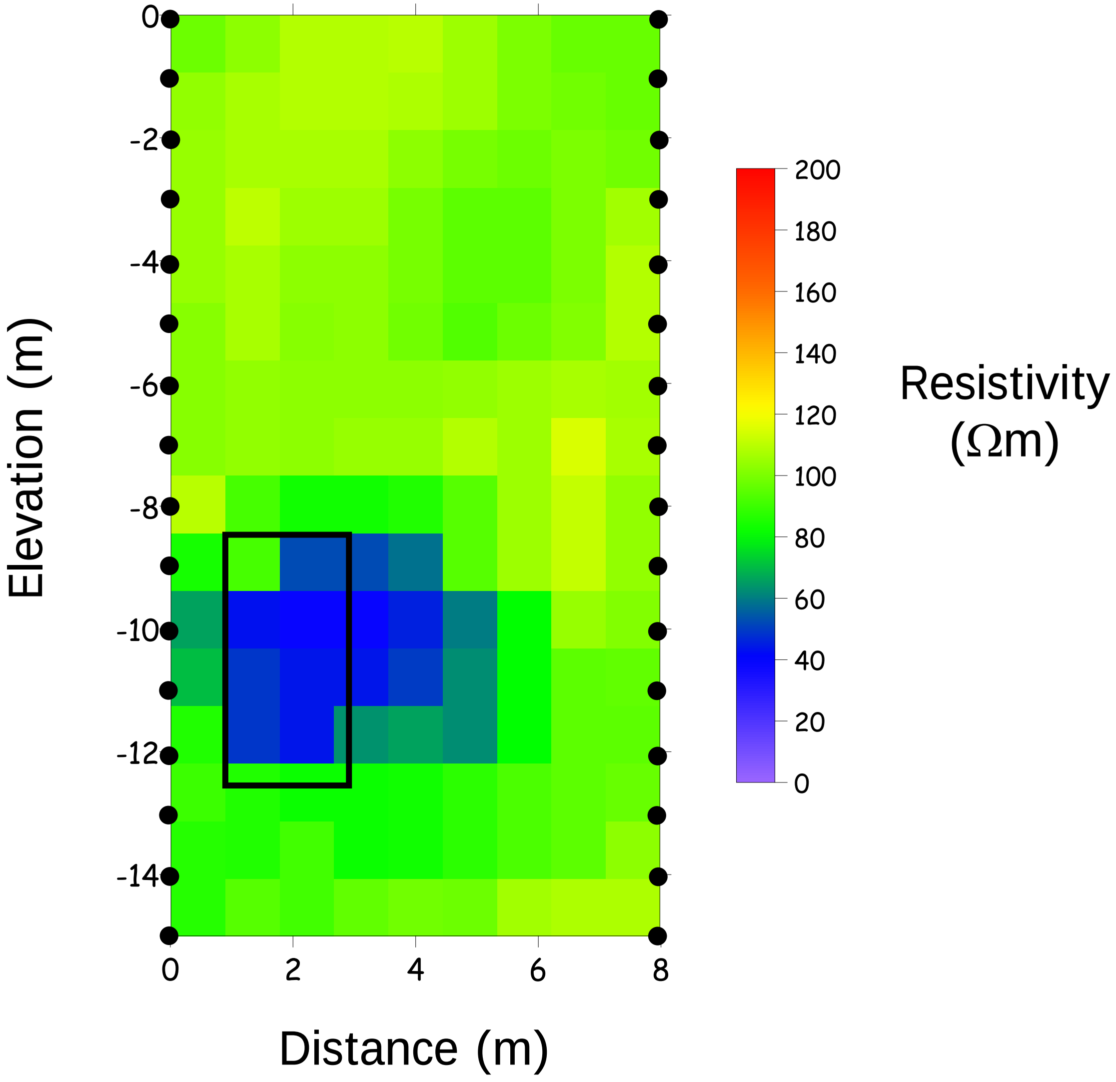
A good rule of thumb is to keep the borehole separation less than 75% of the total electrode array length, i.e. a vertical to horizontal aspect of 1.5

How does the regularisation anisotropy affect the image ?

Inversion of ‘noise free’ data using *Skip 7*

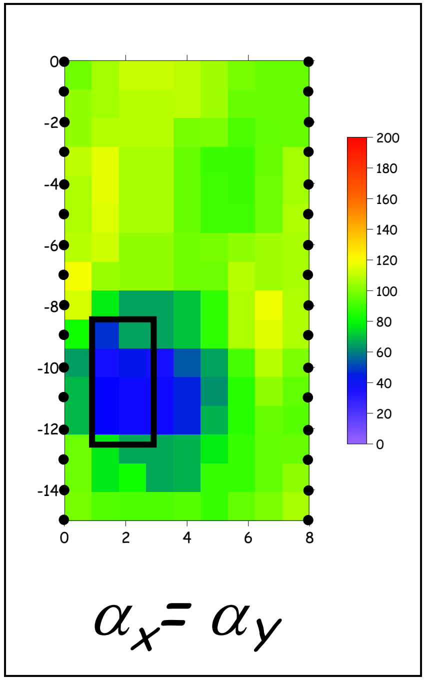
($\alpha_x = 20 \times \alpha_y$)

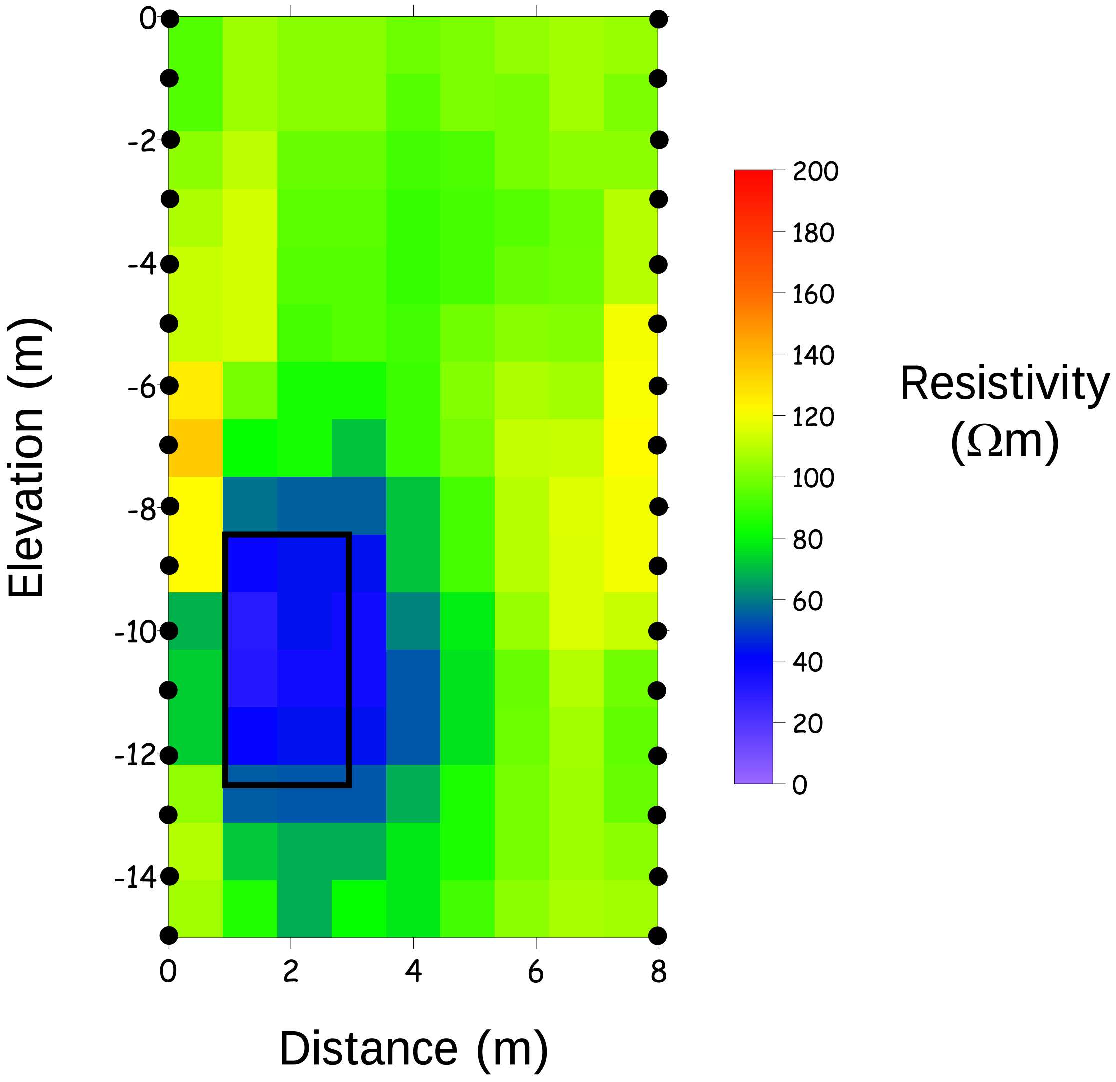




Inversion of 'noise free' data using *Skip 7*

($\alpha_x = 0.05 \times \alpha_y$)





How does we invert time lapse data ?

Individual images corresponding to different times can be imaged and differenced, alternatively:

Use a background image $\mathbf{m_0}$ as a reference within the penalty function, or`

Invert a combined (ratio) dataset …

If we have two datasets $\mathbf{d_t}$ and $\mathbf{d_0}$ then we can compute a combined (ratio) dataset from:

$$\mathbf{d}_r = \frac{\mathbf{d}_t}{\mathbf{d}_0} F(\sigma_{\mathrm{hom}})$$

where σ_{hom} is an arbitrarily chosen conductivity.

The inverted image will then show any changes relative to this reference value.

This *ratio* approach has been used widely and has been essential in some cases for 2-D imaging where 3-D effects are not accounted for in the model.

Resistivity Image appraisal

How do we take into account varying sensitivity in the image ?

For the inversion model used here, the *resolution matrix* $\mathbf{R}$ is formally defined as:

$$\mathbf{m} = \mathbf{R}\mathbf{m}_{true}$$

Where $\mathbf{m}_{true}$ is the true parameter vector

Which, for our solution, equates to:

$$\mathbf{R} = \left(\mathbf{J}_k{}^T \mathbf{W}_\mathrm{d}{}^T \mathbf{W}_\mathrm{d} \mathbf{J}_k + \alpha \mathbf{W}_\mathrm{m}{}^\mathrm{T} \mathbf{W}_\mathrm{m}\right)^{-1} \mathbf{J}_k{}^T \mathbf{W}_\mathrm{d}{}^T \mathbf{W}_\mathrm{d} \mathbf{J}_k$$

$$\mathbf{m} = \mathbf{R}\mathbf{m}_{true}$$

$$\mathbf{R} = \left(\mathbf{J}_k^{\,T}\mathbf{W}_d^{\,T}\mathbf{W}_d\,\mathbf{J}_k + \alpha\mathbf{W}_m^{\,\mathrm{T}}\mathbf{W}_m\right)^{-1}\mathbf{J}_k^{\,T}\mathbf{W}_d^{\,T}\mathbf{W}_d\,\mathbf{J}_k$$

So, in an ideal case **R** will be:

$$\begin{bmatrix} 1 & 0 & 0 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ 0 & 0 & 1 & .. & 0 \\ : & : & : & \ddots & 0 \\ 0 & 0 & 0 & .. & 1 \end{bmatrix}$$

$$\mathbf{R} = \left(\mathbf{J}_k{}^T \mathbf{W}_\mathrm{d}{}^T \mathbf{W}_\mathrm{d} \mathbf{J}_k + \alpha \mathbf{W}_\mathrm{m}{}^\mathrm{T} \mathbf{W}_\mathrm{m}\right)^{-1} \mathbf{J}_k{}^T \mathbf{W}_\mathrm{d}{}^T \mathbf{W}_\mathrm{d} \mathbf{J}_k$$

Any deviation from the identity matrix indicates the effect of smoothing and lack of sensitivity.

Normally the diagonal of $\mathbf{R}$ is shown as an image, on a scale 0 to 1.

One drawback with the resolution matrix is the computational effort needed to form it.

Consequently few reported ERT applications have shown values of **R**

There are alternatives …

Park and Van (1991) used the easily computed matrix product:

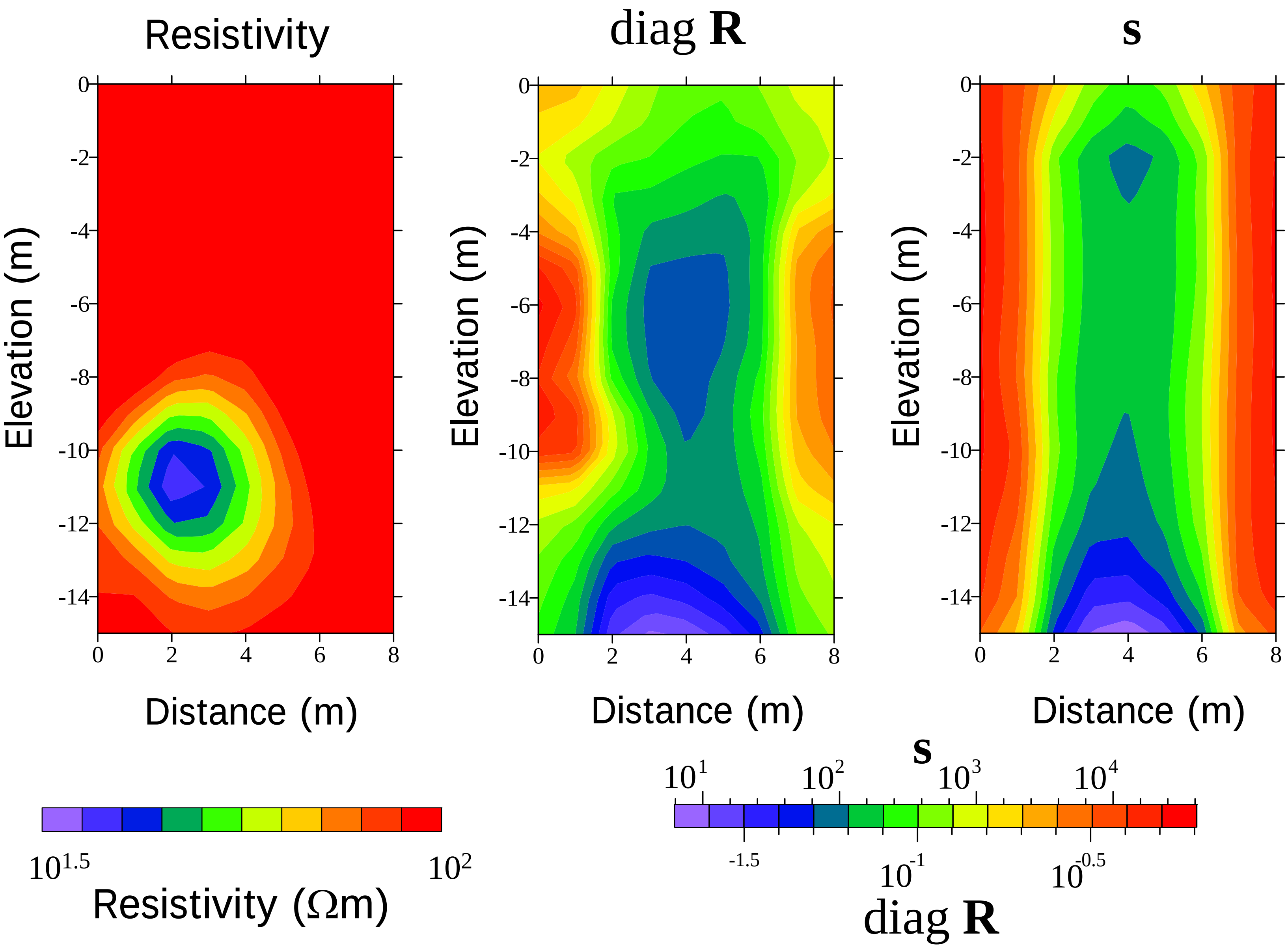
$$\mathbf{J}_k^{\,T}\mathbf{J}_k$$

Kemna (2000) used a similar quantity for his 2-D analysis:

$$\mathbf{s} = \left(\mathbf{J}_k^{\,T}\mathbf{W}_\mathrm{d}^{\,T}\mathbf{W}_\mathrm{d}\mathbf{J}_k\right)$$

Here, vector $\mathbf{s}$ will have large values where sensitivity is high.

Example:



Resistivity
diag R
s
Elevation (m)
Distance (m)
0
-2
-4
-6
-8
-10
-12
-14
0
2
4
6
8
$10^{1.5}$
10^{2}
Resistivity (Ωm)
s
10^{1}
10^{2}
10^{3}
10^{4}
-1.5
10^{-1}
$10^{-0.5}$
diag R

An alternative approach is the *Depth of Investigation* (DOI) index proposed by Oldenburg and Li (1999) for surface imaging.

$$DOI = \frac{\left| m_1 - m_2 \right|}{\left| m_{ref\,1} - m_{ref\,2} \right|} \qquad 0 \le DOI \le 1$$

DOI is computed for each parameter block by performing two inversions with different *reference* parameter values

$$DOI = \frac{\left| m_1 - m_2 \right|}{\left| m_{ref\,1} - m_{ref\,2} \right|} \qquad 0 \le DOI \le 1$$

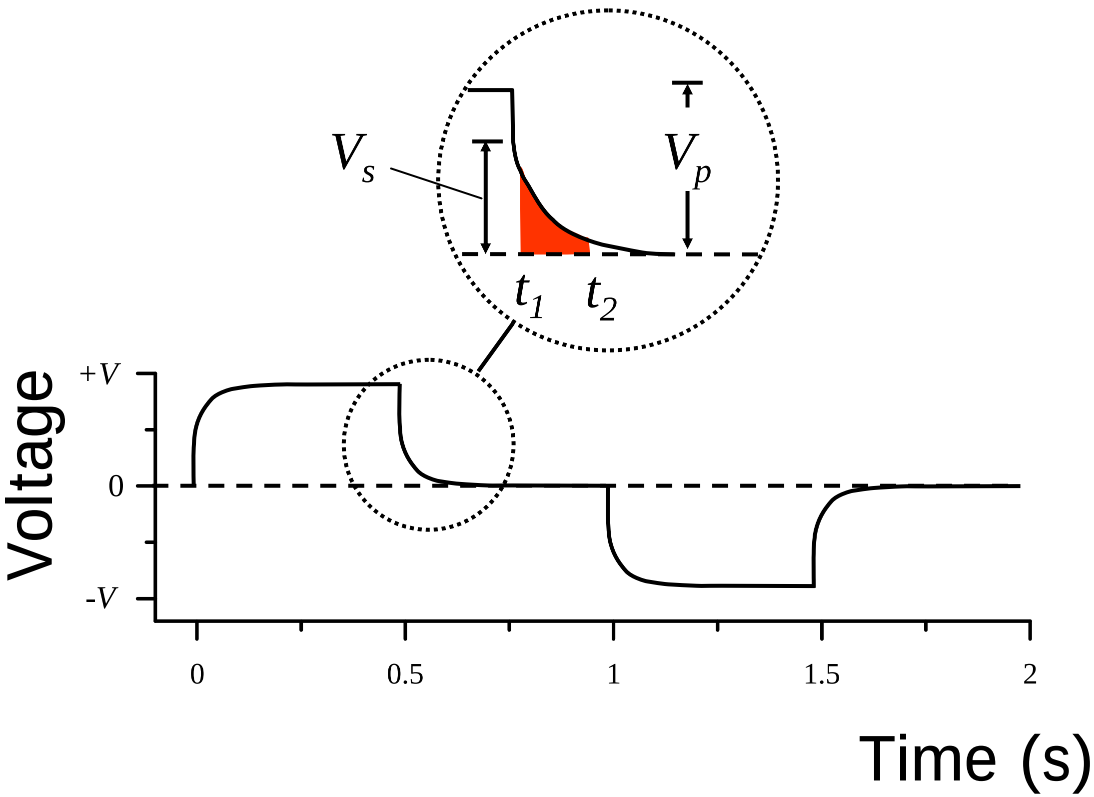
In areas of the imaged region where sensitivity is poor the parameter blocks will change very little from the reference values and the DOI value will be close to unity.

DOI will be close to zero where sensitivity is high.

IP Forward modelling

Recall that the *apparent chargeability is defined* as: $m_a = \frac{V_s}{V_p}$

V_p is the primary voltage and V_s is the secondary voltage



IP Forward modelling

The *apparent chargeability* m_a (measurement) can be obtained from DC resistivity models (Oldenburg & Li, 1994):

$$m_a = \frac{F[(1-m)\sigma_{DC}] - F[\sigma_{DC}]}{F[(1-m)\sigma_{DC}]}$$

where F is the forward operator

Alternatively we can set up the conductivity as a complex variable σ^* (see Binley & Kemna, 2005):

$$\frac{\partial}{\partial x}\left(\sigma^* \frac{\partial V^*}{\partial x}\right) + \frac{\partial}{\partial y}\left(\sigma^* \frac{\partial V^*}{\partial y}\right) + \frac{\partial}{\partial z}\left(\sigma^* \frac{\partial V^*}{\partial z}\right) = -I\,\delta(x)\,\delta(y)\delta(z)$$

IP Inverse modelling

The *apparent chargeability* $m_{a,i}$ of each cell i can be computed from (Oldenburg & Li, 1994):

$$m_{a,i} = \sum_{j=1}^{M} m_j J_{i,j}^{m}$$

where

Voltage for measurement i

$$J_{i,j}^{m} = \frac{\partial \ln V^{i}(\sigma)}{\partial \ln \sigma_j}$$

Conductivity of cell j

This is computationally easy as J is actually formed for DC resistivity inverse modelling

IP Inverse modelling

Alternatively we can solve:

$$\frac{\partial}{\partial x}\left(\sigma^{*}\frac{\partial V^{*}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\sigma^{*}\frac{\partial V^{*}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\sigma^{*}\frac{\partial V^{*}}{\partial z}\right)=-I\,\delta(x)\,\delta(y)\delta(z)$$

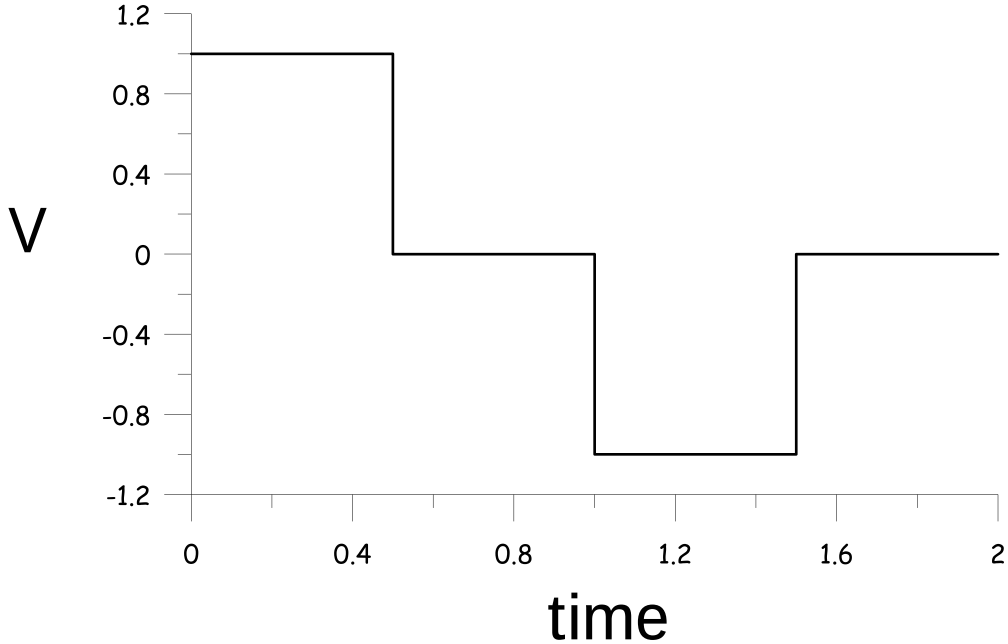
Using the same procedures for DC resistivity (but using complex mathematics) to determine cell parameters σ^{*}

Practical Application guidelines

Here are a few guidelines that will increase the chance of success with ERT for your project ...

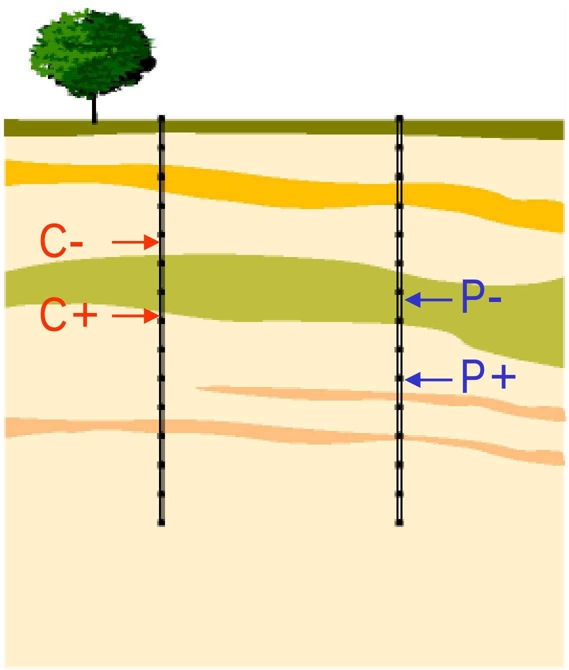
1. Look at the data

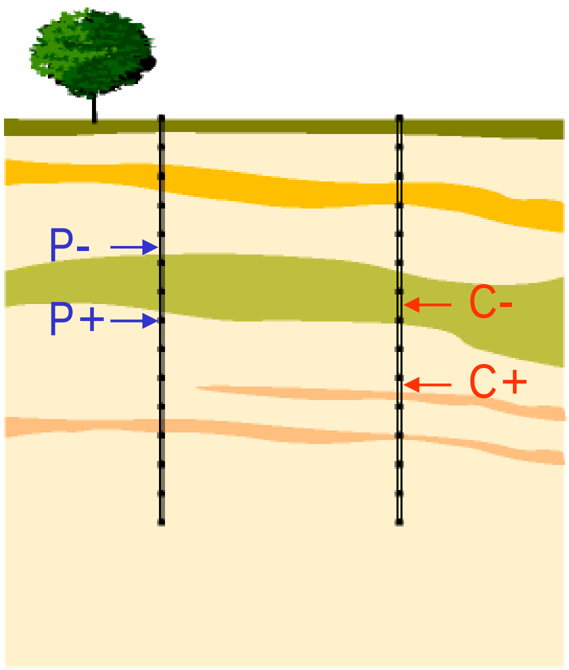
Look at the current and voltage traces on all your measurements and check for departures from the expected/ideal case.



2. Assess data errors

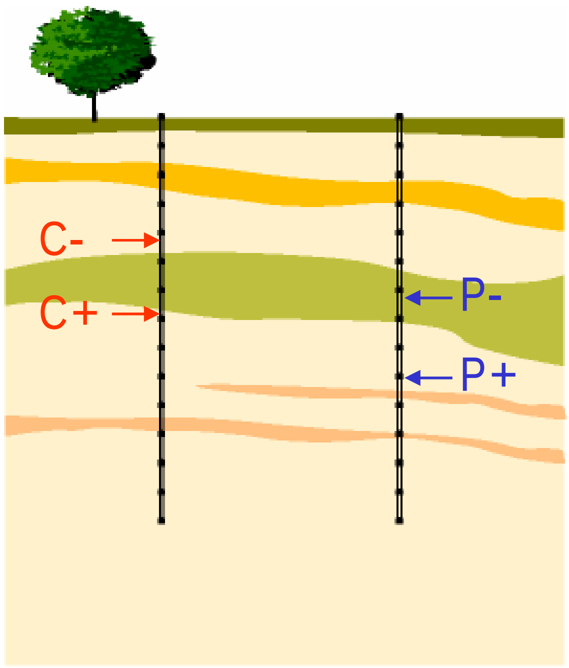
You must determine the errors in the field data. Repeatability may not be suitable for this. From our experience the best way to assess errors is to carry out reciprocal check of ALL measurements.

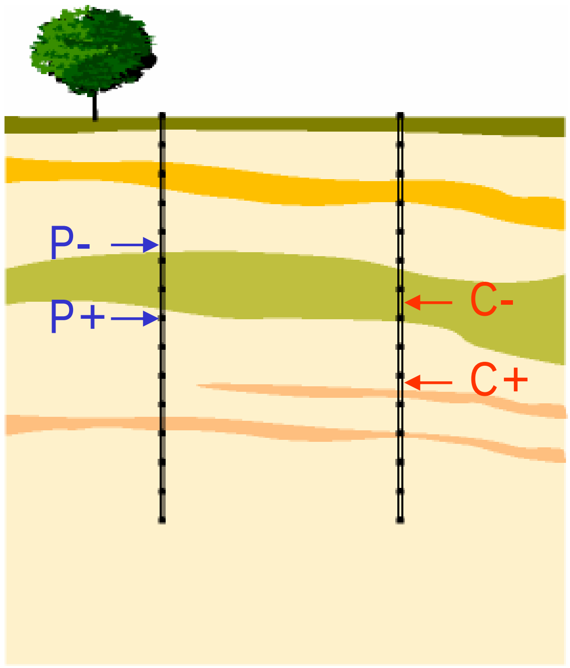




2. Assess data errors

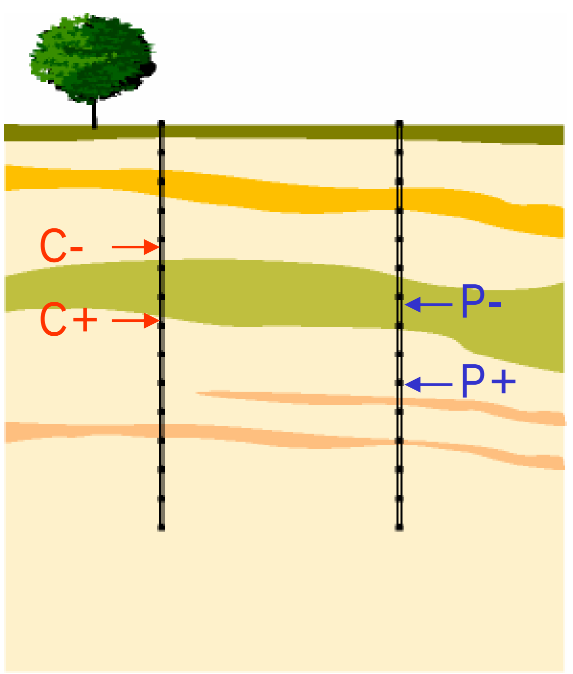
If you have many measurements that do not reciprocate well (say to 5%) then find out why. It is likely to be due to low voltages. If so then decide of your measurement scheme is appropriate.

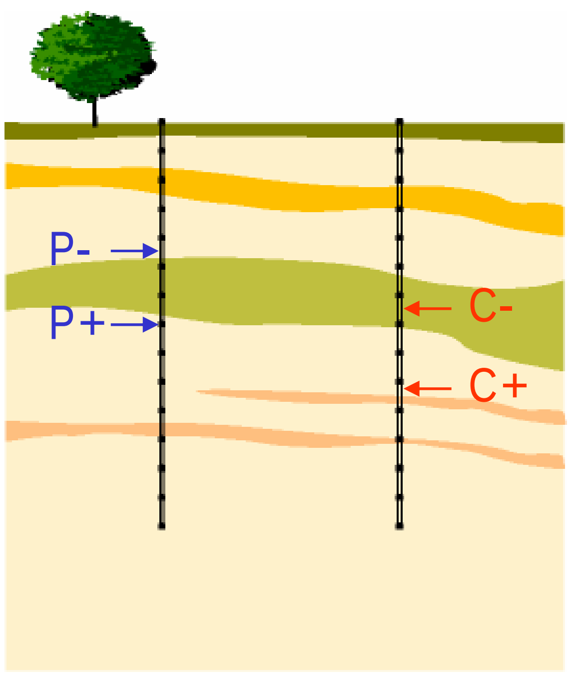




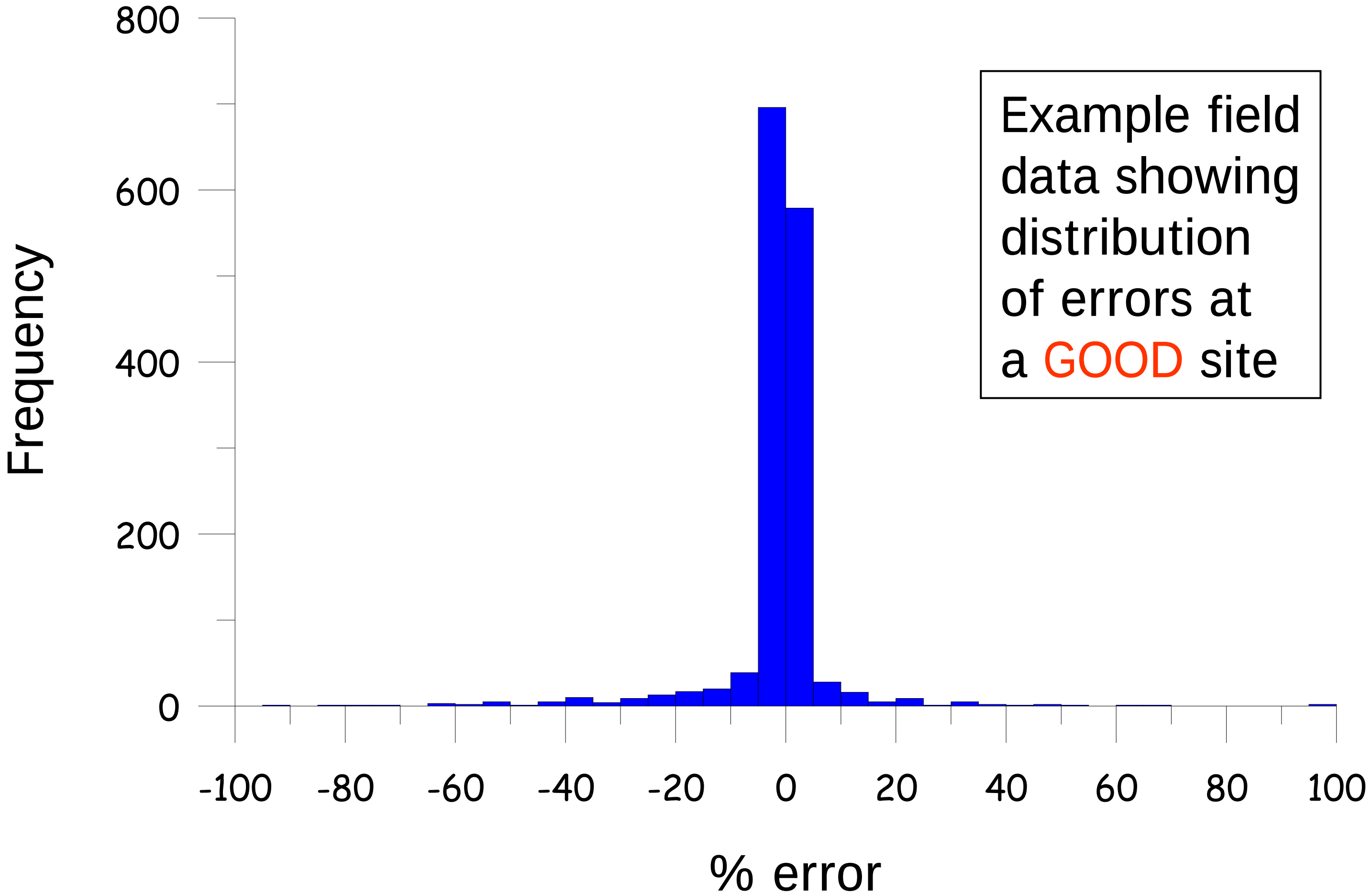
2. Assess data errors

Remove all measurements that do not reciprocate well prior to inversion and use the errors as weights in the inversion.





2. Assess data errors



3. Assess model errors

Make sure you understand how good your forward solver is. It is pointless trying to invert data that is good to 1% if your model is good to only 5%.

Check that you are accounting for any 3-D effects, e.g. borehole effects.

4. Study different measurement schemes

Don’t adopt a favourite scheme but study how other schemes may be more suitable for your problem.

You will have to study modelling errors, likely voltages, time to collect data before carrying out a trial survey to assess field errors.

4. Study different measurement schemes

A number synthetic modelling studies have looked at the relative merits of the different schemes.

We believe that there is no ‘universal’ scheme for ERT.

The optimum scheme will depend on:

- measurement errors (site specific),
- the resistivity structure (site specific),
- resolution required (problem specific),
- data acquisition speed required (problem specific)

24 years ago Lytle and Dines (1978) stated research and development goals for ERT that are relevant even today. They noted in their pioneering paper on the "impedance camera"

"Items worthy of future research include an assessment of the influence of noise in the data, a study of the accuracy of the reconstruction and its spatial dependence, an evaluation of the degree of dependence of various measurement configurations, an analytic study of the resolution limit, and a determination of the extent to which the use of a priori knowledge affects the interpretation"

Many of these items are still the subject of research.